1. The moment-generating function and the central limit theorem.

Suppose that $x$ is a random variable taking values on the real line, and $p(x)$ is a probability distribution for $x$. We say that

$$X_n \equiv \langle x^n \rangle = \int_{-\infty}^{\infty} dx \ p(x)x^n$$

is the $n$th moment of the probability distribution, and that

$$\bar{X}(t) = \langle e^{tx} \rangle = \sum_{n=0}^{\infty} \frac{X_n t^n}{n!}$$

is the moment-generating function of the distribution.

a) Compute the moment-generating function for the normalized Gaussian distribution with mean zero and variance $\sigma^2$,

$$q(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma^2}. \quad (1)$$

(Note that it is easy to do the integral $\bar{X}(t) = \int_{-\infty}^{\infty} dx \ q(x)e^{tx}$ by shifting the integration variable by a constant.)

b) By expanding $\bar{X}(t)$ in a power series, show that for the normalized Gaussian distribution $\langle x^n \rangle = 0$ for $n$ odd, and find an expression for $\langle x^{2n} \rangle$ for each nonnegative integer $n$.

c) Now suppose that $\{x_1, x_2, x_3, \ldots, x_N\}$ are independent random variables, all identically distributed with probability distribution $p(x)$. Consider the random variable

$$u_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i,$$
which (aside from the non-standard but conveniently chosen normalization), represents the result of sampling the same distribution \( N \) times and averaging the results. The moment generating function for \( u_N \) is

\[
\bar{U}_N(t) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \ p(x_1)p(x_2)\cdots p(x_N) \ e^{tu_N};
\]

express \( \bar{U}_N(t) \) in terms of \( \bar{X} \), the moment generating function for the distribution \( p(x) \).

d) Assuming that the distribution \( p(x) \) has mean zero (\( \langle x \rangle = 0 \)), show that \( \bar{U}_N(t) \) can be approximated as

\[
\bar{U}_N(t) \approx \left( 1 + \frac{t^2}{2N} X_2 + O \left( N^{-3/2} \right) \right)^N,
\]

and show that in the limit \( N \to \infty \), \( \bar{U}_N(t) \) becomes the moment-generating function of a Gaussian distribution with mean zero. What is the variance of this Gaussian?

2. Biased coin. When a biased coin is flipped the outcome is heads with probability \( p \) and tails with probability \( 1 - p \). If this coin is flipped \( N \) times, the probability that the total number of heads is \( n \) is

\[
p(n) = \binom{N}{n} p^n (1 - p)^{N-n}.
\]

The most likely value of \( n \) is \( n = pN \), but there are fluctuations about this most likely value.

Denote \( n = Np + s \), and suppose that \( N \gg 1 \). In this limit, \( p(n) \), regarded as a function of \( s \), approaches a Gaussian with mean zero and some variance \( \sigma_p^2 \); hence,

\[
\ln[p(n)] = \text{constant} - \frac{s^2}{2\sigma_p^2} + O \left( s^4 \right),
\]

where “constant” means a term independent of \( s \). Calculate \( \sigma_p^2 \) using the Stirling approximation and the approximations \( s \ll pN \) and \( s \ll (1-p)N \). To save work, notice that you only need to find the coefficient of \( s^2 \) in the expansion of \( \ln[p(n)] \); you don’t need to worry about the constant terms or the linear terms. Compare your value of \( \sigma_p^2 \) with the result \( \sigma^2 = N/4 \) found in class for the case \( p = 1/2 \).
3. **Probability of a large deviation.** For the Gaussian distribution Eq. (1), $x$ is not likely to deviate from zero by an amount much larger than $\sigma$. To estimate the probability of a large deviation, we observe that the probability for $x$ to have a value exceeding $t$,

$$P(x \geq t) = \int_t^\infty dx \, q(x),$$

has an asymptotic expansion for $t^2 \gg \sigma^2$.

a) Integrate by parts repeatedly to show that

$$P(x \geq t) = \sqrt{\frac{\sigma^2}{2\pi t^2}} \, e^{-t^2/2\sigma^2} \left( A - B \frac{\sigma^2}{t^2} + O \left( \frac{\sigma^4}{t^4} \right) \right),$$

where $A$ and $B$ are positive constants, and find $A$ and $B$.

b) Estimate the probability that $x$ is $10\sigma$ or larger.