Problem 1
At time \( t = 0 \) a particle in the potential \( V(x) = m\omega^2x^2/2 \) is described by the wave function
\[
\psi(x, 0) = A \sum_n (1/\sqrt{2})^n \psi_n(x),
\]
where \( \psi_n(x) \) are eigenstates of the energy with eigenvalues \( E_n = (n + 1/2)\hbar\omega \).

You are given that \( (\psi_n, \psi_n') = \delta_{nn'} \).

(a) Find the normalization constant \( A \).

(b) Write an expression for \( \psi(x, t) \) for \( t > 0 \).

(c) Show that \( |\psi(x, t)|^2 \) is a periodic function of time and indicate the longest period \( \tau \).

(d) Find the expectation value of the energy at \( t = 0 \).

(Hint: You don’t need to know the explicit wave functions \( \psi_n(x) \).)

Problem 2
(Exercise 5.2.1) A particle is in the ground state of a box of length \( L \). Suddenly the box expands (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is \((8/3\pi)^2\).

Problem 3
(Exercise 5.2.3) Consider \( V(x) = -aV_0\delta(x) \). Show that it admits a bound state of energy \( E = -ma^2V_0^2/2\hbar^2 \). Are there any other bound states?

(Hint: Solve Schrödinger’s equation outside the potential for \( E < 0 \), and keep only the solution that has the right behavior at infinity and is continuous at \( x = 0 \). Draw the wave function and see how there is a cusp, or a discontinuous change of slope at \( x = 0 \). Calculate the change in slope and equate it to
\[
\int_{-\varepsilon}^{+\varepsilon} \left( \frac{d^2\psi}{dx^2} \right) dx
\]
(where \( \varepsilon \) is infinitesimal) determined from Schrödinger’s equation.)
Problem 4

(Exercise 5.2.6) Consider a particle in a square well potential:

\[ V(x) = \begin{cases} 0, & |x| \leq a \\ V_0, & |x| \geq a \end{cases} \]

Since when \( V_0 \to \infty \), we have a box, let us guess what the lowering of the walls does to the states. First of all, all the bound states (which alone we are interested in), will have \( E \leq V_0 \). Second, the wave functions of the low-lying levels will look like those of the particle in a box with the obvious difference that \( \psi(x) \) will not vanish at the walls but instead spill out with an exponential tail. The eigenfunctions will still be even, odd, even, etc.

(1) Show that the even solutions have energies that satisfy the transcendental equation

\[ k \tan ka = \kappa \]  

(1)

while the odd ones will have energies that satisfy

\[ k \cot ka = -\kappa \]  

(2)

where \( k \) and \( i\kappa \) are the real and complex wave numbers inside and outside the well, respectively. Note that \( k \) and \( \kappa \) are related by

\[ k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2} \]  

(3)

Verify that as \( V_0 \) tends to \( \infty \), we regain the levels in the box

(2) Equations (1) and (2) must be solved graphically. In the \((\alpha = ka, \beta = \kappa a)\) plane, imagine a circle that obeys Eq. (3). The bound states are then given by the intersection of the curve \( \alpha \tan \alpha = \beta \) or \( \alpha \cot \alpha = -\beta \) with the circle. (Remember \( \alpha \) and \( \beta \) are positive.)

(3) Show that there is always one even solution and that there is no odd solution unless \( V_0 \geq \hbar^2 \pi^2 / 8ma^2 \). What is \( E \) when \( V_0 \) just meets this requirement?