



Quantum information and precision measurement

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Abstract. We describe some applications of quantum information theory to the analysis of quantum limits on measurement sensitivity. A measurement of a weak force acting on a quantum system is a determination of a classical parameter appearing in the master equation that governs the evolution of the system; limitations on measurement accuracy arise because it is not possible to distinguish perfectly among the different possible values of this parameter. Tools developed in the study of quantum information and computation can be exploited to improve the precision of physics experiments; examples include superdense coding, fast database search, and the quantum Fourier transform.

1. Introduction: distinguishability of superoperators

The exciting recent developments in the theory of quantum information and computation have already established an enduring legacy. The two most far-reaching results—that a quantum computer (apparently) can solve problems that will forever be beyond the reach of classical computers [1], and that quantum information can be protected from errors if properly encoded [2]—have surely earned a prominent place at the foundations of computer science.

The implications of these ideas for the future of physics are less clear, but we expect them to be profound. In particular, we anticipate that our deepening understanding of quantum information will lead to new strategies for pushing back the boundaries of quantum-limited measurements. Quantum entanglement, quantum error correction, and quantum information processing can all be exploited to improve the information-gathering capability of physics experiments.

In a typical quantum-limited measurement, a classical signal is conveyed over a quantum channel [3]. Nature sends us a message, such as the value of a weak force, that can be regarded as a classical parameter appearing in the Hamiltonian of the apparatus (or more properly, if there is noise, its master equation). The apparatus undergoes a quantum operation $\mathcal{S}(a)$, and we are to extract as much information as we can about the parameter(s) a by choosing an initial preparation of the apparatus, and a positive-operator-valued measure (POVM) to read it out. Quantum information theory should be able to provide a theory of the *distinguish-*

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ability of superoperators, a measure of how much information we can extract that distinguishes one superoperator from another, given some specified resources that are available for the purpose. This distinguishability measure would characterize the inviolable limits on measurement precision that can be achieved with fixed resources.

Many applications of quantum information theory involve the problem of distinguishing non-orthogonal quantum *states*. For example, a density operator ρ_a is chosen at random from an ensemble $\mathcal{E} = \{\rho_a, p_a\}$ (where p_a is an *a priori* probability), and a measurement is performed to extract information about which ρ_a was chosen. The problem of distinguishing *superoperators* is rather different, but the two problems are related. For example, let us at first ignore noise, and also suppose that the classical force we are trying to detect is static. Then we are trying to identify a particular time-independent Hamiltonian H_a that has been drawn from an ensemble $\{H_a, p_a\}$. We may choose a particular initial pure state $|\psi_0\rangle$, and then allow the state to evolve, as governed by the unknown Hamiltonian, for a time t ; our ensemble of possible Hamiltonians generates an ensemble of pure states

$$\{|\psi_a(t)\rangle = \exp(-itH_a)|\psi_0\rangle, p_a\}. \quad (1)$$

Since our goal is to gain as much information as possible about the applied Hamiltonian, we should choose the initial state $|\psi_0\rangle$ so that the resulting final states are maximally distinguishable.

There are many variations on the problem, distinguished in part by the resources we regard as most valuable. We might have the freedom to choose the elapsed time as we please, or we might impose constraints on t . We might have the freedom to modify the Hamiltonian by adding an additional ‘driving’ term that is under our control. We might use an *adaptive* strategy, where we make repeated (possibly weak) measurements, and our choice of initial state or driving term in later measurements takes into account the information already collected in earlier measurements [4].

Imposing an appropriate cost function on resources is an important aspect of the formulation of the problem, particularly in the case of the detection of a static (DC) signal. For example, we could in principle repeat the measurement procedure many times to continually improve the accuracy of our estimate. In this respect, the problem of distinguishing superoperators does not have quite so fundamental a character as the problem of distinguishing states, as in the latter case the no-cloning principle [5] prevents us from making repeated measurements on multiple copies of the unknown state. But for a time-dependent signal that stays ‘on’ for a finite duration, there will be a well-defined notion of the optimal strategy for distinguishing one possible signal from another, once our apparatus and its coupling to the classical signal have been specified. Still, for the sake of simplicity, we will mostly confine our attention here to the case of DC signals.

We do not know exactly what shape this nascent theory of the distinguishability of superoperators should take, but we hope that further research can promote the development of new strategies for performing high-precision measurements. On the one hand we envision a programme of research that will be relevant to real laboratory situations. On the other hand, we seek results that are to some degree robust and general (not tied to some particular model of decoherence, or to a particular type of coupling between quantum probe and classical signal). Natu-

rally, there is some tension between these two central desiderata; rather than focus on a specific experimental context, we lean here toward more abstract formulations of the problem.

Our discussion is far from definitive; its goal is to invite a broader community to consider these issues. We will mostly be content to observe that some familiar concepts from the theory of quantum information and computation can be translated into tools for the measurement of classical forces. Some examples include superdense coding, fast database search, and the quantum Fourier transform.

Naturally, the connections between quantum information theory and precision measurement have been recognized previously by many authors. Especially relevant is the work by Helstrom [6], by Wootters [7], by Braunstein [8], and by Braunstein and Caves [9] on state distinguishability and parameter estimation, and by Braginsky and others [10] on quantum non-demolition measurement. The problem of characterizing an unknown superoperator has been studied by Poyatos *et al.* [11], by Chuang and Nielsen [12], and by Bužek [13]. Optimal measurements for distinguishing unitary transformations have been analysed by Derka *et al.* [14], based on earlier work by Massar and Popescu [15]. Optimal strategies for the measurement of frequency and time have been discussed by Huelga *et al.* [16] and by Bužek *et al.* [17]. That entanglement can be exploited to improve the information extracted in a measurement is also highlighted by the interesting recent work of Gisin and Popescu [18].

Although what we have to add may be relatively modest, we hope that it may lead to further progress.

2. Superdense coding: improved distinguishability through entanglement

Recurring themes of quantum information theory are that entanglement can be a valuable resource, and that entangled measurements sometimes can collect more information than unentangled measurements. It should not be surprising, then, if the experimental physicist finds that the best strategies for detecting a weak classical signal involve the preparation of entangled states and the measurement of entangled observables.

Suppose, for example, that our apparatus is a single-qubit, whose time-independent Hamiltonian (aside from an irrelevant additive constant), can be expressed as

$$H_{\mathbf{a}} = \mathbf{a} \cdot \boldsymbol{\sigma} ; \quad (2)$$

here $\mathbf{a} = (a_1, a_2, a_3)$ is an unknown three-vector, and $\sigma_{1,2,3}$ are the Pauli matrices. (We may imagine that a spin- $\frac{1}{2}$ particle with a magnetic moment is employed to measure a static magnetic field.) By preparing an initial state of the qubit, allowing the qubit to evolve, and then performing a single measurement, we can extract at best one bit of information about the magnetic field (as Holevo's theorem [19] ensures that the optimal POVM in a two-dimensional Hilbert space can acquire at most one bit of information about a quantum state).

If we have two qubits, and measure them one at a time, we can collect at best two bits of information about the magnetic field. In principle, this could be enough to distinguish perfectly among four possible values of the field. In practice, for a

generic choice of four Hamiltonians labelled by vectors $\mathbf{a}^{(1,2,3,4)}$, the optimal information gain cannot be achieved by measuring the qubits one at a time. Rather a better strategy exploits quantum entanglement.

An improved strategy can be formulated by following the paradigm of superdense coding [20], whereby shared entanglement is exploited to enhance classical communication between two parties. To implement superdense coding, the sender (Alice) and the receiver (Bob) use a shared Bell state

$$|\phi^+\rangle = 2^{-1/2}(|00\rangle + |11\rangle) \tag{3}$$

that they have prepared previously. Alice applies one of the four unitary operators $\{I, \sigma_1, \sigma_2, \sigma_3\}$ to her member of the entangled pair, and then sends it to Bob. Upon receipt, Bob possesses one of the four mutually orthogonal Bell states

$$\begin{aligned} |\phi^+\rangle &= 2^{-1/2}(|00\rangle + |11\rangle) = I \otimes I |\phi^+\rangle, \\ |\psi^+\rangle &= 2^{-1/2}(|01\rangle + |10\rangle) = \sigma_1 \otimes I |\phi^+\rangle, \\ -i|\psi^-\rangle &= \frac{-i}{2^{1/2}}(|01\rangle - |10\rangle) = \sigma_2 \otimes I |\phi^+\rangle, \\ |\phi^-\rangle &= 2^{-1/2}(|00\rangle - |11\rangle) = \sigma_3 \otimes I |\phi^+\rangle; \end{aligned} \tag{4}$$

by performing an entangled Bell measurement (simultaneous measurements of the commuting collective observables $\sigma_1 \otimes \sigma_1$ and $\sigma_3 \otimes \sigma_3$), Bob can perfectly distinguish the states. Although only one qubit passes from Alice to Bob, two classical bits of information are transmitted and successfully decoded. In fact, this enhancement of the transmission rate is optimal—with shared entanglement, no more than two classical bits can be carried by each transmitted qubit [21].

The lesson of superdense coding is that entanglement can allow us to better distinguish operations on quantum states, and we may apply this method to the problem of distinguishing Hamiltonians†. Let us imagine that the magnitude of the magnetic field is known, but not its direction—then we can choose our unit of time so that $|\mathbf{a}| = 1$. We may prepare a pair of qubits in the entangled state $|\phi^+\rangle$, and expose only one member of the pair to the magnetic field while the other remains well shielded. In time t , the state evolves to

$$\begin{aligned} |\psi_{\hat{a}}(t)\rangle &\equiv \exp(-itH_{\hat{a}} \otimes I)|\phi^+\rangle \\ &= [\cos t(I \otimes I) - i \sin t(\hat{a} \cdot \boldsymbol{\sigma} \otimes I)]|\phi^+\rangle \\ &= \cos t|\phi^+\rangle \\ &\quad - i \sin t[a_1|\psi^+\rangle - ia_2|\psi^-\rangle + a_3|\phi^+\rangle]; \end{aligned} \tag{5}$$

the inner product between the states arising from Hamiltonians $H_{\hat{a}}$ and $H_{\hat{b}}$ becomes

$$\langle \psi_{\hat{a}}(t) | \psi_{\hat{b}}(t) \rangle = \cos^2 t + (\hat{a} \cdot \hat{b}) \sin^2 t. \tag{6}$$

For these states to be orthogonal, we require

† This idea was suggested to us by Chris Fuchs [22].

$$\hat{a} \cdot \hat{b} = -\cot^2 t. \quad (7)$$

Since $\cot^2 t \geq 0$, the states are not orthogonal for any value of t unless the two magnetic field directions \hat{a} and \hat{b} are separated by at least 90° .

Now suppose that the magnetic field (of known magnitude) points in one of three directions that are related by three-fold rotational symmetry. These directions could form a planar trine with $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = \hat{b} \cdot \hat{c} = -1/2$, or a 'lifted trine' with angle θ between each pair of directions, where $-1/2 \leq \cos \theta \leq 0$. For any such trine of field directions, we may evolve for a time t such that

$$\cot^2 t = -\cos \theta, \quad (8)$$

and perform an (entangled) orthogonal measurement to determine the field. At the point of tetrahedral symmetry, $\cos \theta = -1/3$, we may add a fourth field direction such that the inner product for each pair of field directions is $-1/3$; then all four directions can be perfectly distinguished by Bell measurement.

In this case of four field directions with tetrahedral symmetry, the two-bit measurement outcome achieves a two-bit information gain, if the four directions were equally likely *a priori*. In contrast, no adaptive strategy in which single qubits are measured one at a time can attain a two-bit information gain. This separation between the information gain attainable through entangled measurement and that attainable through adaptive non-entangled measurement, for the problem of distinguishing Hamiltonians, recalls the analogous separation noted by Peres and Wootters [23] for the problem of distinguishing non-orthogonal states.

3. Grover's database search: improved distinguishability through driving

Another instructive example is Grover's method [24] for searching an unsorted database, which (as formulated by Farhi and Gutmann [25]) we may interpret as a method for improving the distinguishability of a set of Hamiltonians by adding a controlled driving term.

Consider an N -dimensional Hilbert space with orthonormal basis $\{|x\rangle\}$, $x = 0, 1, 2, \dots, N-1$, and suppose that the Hamiltonian for this system is known to be one of the N operators

$$H_x = E|x\rangle\langle x|. \quad (9)$$

We are to perform an experiment that will allow us to estimate the value of x .

We could, for example, prepare the initial state $2^{-1/2}(|y\rangle + |y'\rangle)$, allow the system to evolve for a time $T = \pi/E$, and then perform an orthogonal measurement in the basis $|\pm\rangle = 2^{-1/2}(|y\rangle \pm |y'\rangle)$. Then we will obtain the outcome $|-\rangle$ if and only if one of y, y' is x . Searching for x by this method, we would have to repeat the experiment for $O(N)$ distinct initial states to have any reasonable chance of successfully inferring the value of x .

Our task becomes easier if we are able to modify the Hamiltonian by adding a term that we control to drive the system. We choose the driving term to be

$$H_D = E|s\rangle\langle s|, \quad (10)$$

where $|s\rangle$ denotes the state

$$|s\rangle = \frac{1}{N^{1/2}} \sum_{y=0}^{N-1} |y\rangle. \quad (11)$$

Then the full Hamiltonian is

$$H'_x = H_x + H_D = E(|x\rangle\langle x| + |s\rangle\langle s|), \quad (12)$$

and we can readily verify that the vectors

$$|E_{\pm}\rangle \equiv |s\rangle \pm |x\rangle \quad (13)$$

are (unconventionally normalized!) eigenstates of H with the eigenvalues

$$E_{\pm} = E \left(1 \pm \frac{1}{N^{1/2}} \right). \quad (14)$$

We may prepare the initial state

$$|s\rangle = \frac{1}{2}(|E_+\rangle + |E_-\rangle); \quad (15)$$

since the energy splitting is $\Delta E = 2E/N^{1/2}$, after a time

$$T = \pi/\Delta E = \pi N^{1/2}/2E, \quad (16)$$

this state flops to the state

$$\frac{1}{2}(|E_+\rangle - |E_-\rangle) = |x\rangle. \quad (17)$$

Thus, by performing an orthogonal measurement, we can learn the value of x with certainty [25].

The driving term we have chosen is the continuous time analogue of the iteration employed by Grover [24] for rapid searching. And as the Grover search algorithm can be seen to be optimal, in the sense that a marked state can be identified with high probability with the minimal number of oracle calls [26], so the driving term we have chosen is optimal in the sense that it enables us to identify the value of the classical parameter labelling the Hamiltonian in the minimal time, at least asymptotically for N large. (In a physics experiment, the 'oracle' is Nature, whose secrets we are eager to expose.) For this Grover–Farhi–Gutmann problem, we can make a definite statement about how to optimize expenditure of a valuable resource (time) in the identification of a system Hamiltonian.

We also note that adding a driving term can sometimes improve the efficacy of the superdense coding method described in section 1. For example, in the case of three magnetic fields of equal magnitude with three-fold symmetry, but with an angle between fields of less than 90° , applying a driving field along the line of symmetry can make the resultant field directions perfectly distinguishable. In fact, Beckman [27] has shown that for any three field vectors forming a triangle that is isosceles or nearly isosceles, a suitable driving field can always be found such that the field directions can be distinguished perfectly.

4. Distinguishing two alternatives

Let us consider the special case in which our apparatus is known to be governed by one of two possible Hamiltonians H_1 or H_2 . If the system is two dimensional, we are trying to distinguish two possible values \mathbf{a}, \mathbf{b} of the magnetic field with a

spin- $\frac{1}{2}$ probe. Suppose for simplicity that the two fields have the same magnitude (normalized to unity), but differing directions.

Assuming that we are unable to modify the Hamiltonian by adding a driving term, the optimal strategy is to choose an initial polarization vector that bisects the two field directions \hat{a}, \hat{b} . Depending on the actual value of the field, the polarization will precess on one of two possible cones. If the angle between \hat{a} and \hat{b} is $\theta \geq 90^\circ$, then the two possible polarizations will eventually be back-to-back; an orthogonal measurement performed at that time will distinguish \hat{a} and \hat{b} perfectly. But if $\theta < 90^\circ$, the two polarizations are never back-to-back; the best strategy is to wait until the angle between the polarizations is maximal, and to then perform the orthogonal measurement that best distinguishes them. We cannot perfectly distinguish the two field directions by this method.

On the other hand, if we are able to apply a known driving magnetic field in addition to the unknown field that is to be determined, then two fields \mathbf{a} and \mathbf{b} can always be perfectly distinguished. If we apply the field $-\mathbf{b}$, then the problem is one of distinguishing the trivial Hamiltonian from

$$H_{\text{diff}} = (\mathbf{a} - \mathbf{b}) \cdot \boldsymbol{\sigma}. \quad (18)$$

We can choose an initial polarization orthogonal to $\mathbf{a} - \mathbf{b}$, and wait just long enough for H_{diff} to rotate the polarization by π . Then an orthogonal measurement perfectly distinguishes H_{diff} from the trivial Hamiltonian.

Evidently, the same strategy can be applied to distinguish two Hamiltonians H_1 and H_2 in a Hilbert space of arbitrary dimension. We drive the system with $-H_2$; then to distinguish the trivial Hamiltonian from $H_1 - H_2$, we chose the initial state

$$2^{-1/2}(|E_{\min}\rangle + |E_{\max}\rangle), \quad (19)$$

where E_{\min}, E_{\max} are the minimal and maximal eigenvalues of $H_1 - H_2$. After a time t with

$$t(E_{\max} - E_{\min}) = \pi, \quad (20)$$

this state evolves to the orthogonal state $2^{-1/2}(|E_{\min}\rangle - |E_{\max}\rangle)$, so that the trivial and non-trivial Hamiltonians can be perfectly distinguished.

In the case of the two-dimensional version of the ‘Grover problem’ with $H_1 = |0\rangle\langle 0|$ and $H_2 = |1\rangle\langle 1|$, this choice for the driving Hamiltonian actually outperforms the Grover driving term of equation (10)—the two Hamiltonians can be distinguished in a time that is shorter by a factor of $2^{1/2}$. So while the Grover strategy is optimal for asymptotically large N , it is not actually optimal for $N = 2$.

5. Distinguishing two alternatives in a fixed time

Let us now suppose that we are to distinguish between two time-independent Hamiltonians H_1 and H_2 , and that a *fixed duration* t has been allotted to perform the experiment. Is the driving strategy described above (in which $-H_2$ is added to the Hamiltonian) always the best possible?

If we have the freedom to add a driving term of our choice, then we may assume without loss of generality that we are to distinguish the non-trivial Hamiltonian H from the trivial Hamiltonian 0. As already noted, if the largest

difference $\Delta E = E_{\max} - E_{\min}$ of eigenvalues of H satisfies $t\Delta E \geq \pi$, then H can be perfectly distinguished from 0; let us therefore suppose that $t\Delta E < \pi$.

If we add a *time-independent* driving term K to the Hamiltonian, and choose an initial state $|\psi_0\rangle$, then after a time t , we will need to distinguish the two states

$$\exp(-itK)|\psi_0\rangle, \quad \exp(-it(H+K))|\psi_0\rangle. \quad (21)$$

Two pure states will be more distinguishable when their inner product is smaller. Therefore, to best distinguish $H+K$ from K , we should choose $|\psi_0\rangle$ to minimize the inner product

$$|\langle\psi_0|\exp(itK)\exp[-it(H+K)]|\psi_0\rangle|. \quad (22)$$

If we expand $|\psi_0\rangle$ in terms of the eigenstates $\{|a\rangle\}$ of $\exp(itK)\exp[-it(H+K)]$ with eigenvalues $\{\exp(-itE_a)\}$,

$$|\psi_0\rangle = \sum_a \alpha_a |a\rangle, \quad (23)$$

this inner product becomes

$$|\langle\psi_0|\exp(itK)\exp[-it(H+K)]|\psi_0\rangle| = \left| \sum_a |\alpha_a|^2 \exp(-itE_a) \right|. \quad (24)$$

The right-hand side of equation (24) is the modulus of a convex sum of points on the unit circle. Assuming the modulus is bounded from zero, it attains its minimum when $|\psi_0\rangle$ is the equally weighted superposition of the extremal eigenstates of $\exp(itK)\exp[-it(H+K)]$ —those whose eigenvalues are maximally separated on the unit circle. For $K=0$, the minimum is $\cos(t\Delta E/2)$, where ΔE is the difference of the maximal and minimal eigenvalues of H .

We prove in Appendix A that turning on a non-zero driving term K can never cause the extremal eigenvalues to separate further, and therefore can never improve the distinguishability of the two states in equation (21)[†]. Therefore, $K=0$ is the optimal driving term for distinguishing two Hamiltonians. In other words, if we wish to distinguish between two Hamiltonians H_1 and H_2 , it is always best to turn on a driving term that precisely cancels one of the two.

The above discussion encompasses the strategy of introducing an ancilla entangled with the probe (which proved effective for the problem of distinguishing three or more alternatives). If we wish to distinguish two Hamiltonians $H_1 \otimes I$ and $H_2 \otimes I$ that both act trivially on the ancilla, the optimal driving term exactly cancels one of them (e.g. $K = -H_2 \otimes I$), and so it too acts trivially on the ancilla. We derive no benefit from the ancilla when there are only two alternatives.

Similarly, if we are trying to distinguish only two time-independent signals in an allotted time, it seems likely there is no advantage to performing a sequence of weak measurements, and adapting the driving field in response to the incoming stream of measurement data.

6. More alternatives: adaptive driving

Now suppose that there are N possible Hamiltonians H_1, H_2, \dots, H_N . If there is no time limitation, we can distinguish them perfectly by implementing an

[†] That this might be the case was suggested to us by Fuchs [22].

adaptive procedure; we make a series of measurements, modifying our driving term and initial state in response to the stream of measurement outcomes.

The correct Hamiltonian can be identified by pairwise elimination. First, assume that either H_1 or H_2 is the actual Hamiltonian, and apply a driving term to perfectly distinguish them, say $H_D = -H_1$. After preparing the appropriate initial state and waiting the appropriate time, we make an orthogonal measurement with two outcomes—the result indicates that either H_1 or H_2 is the actual Hamiltonian†. If the result is H_1 , there are two possibilities: either H_1 really is the Hamiltonian, or the assumption that one of H_1 or H_2 is the Hamiltonian was wrong. Either way, H_2 has been eliminated. Similarly, if H_2 is found, H_1 is eliminated. This procedure can then be repeated, eliminating one Hamiltonian per measurement, thereby perfectly distinguishing among the N Hamiltonians in a total of $N - 1$ measurements.

This algorithm is quite inefficient, however. The measurement record is $N - 1$ bits long, while the information gain is only $\log N$ bits.

7. Adaptive phase measurement and the semi-classical quantum Fourier transform

Far more efficient adaptive procedures can be formulated in some cases. Consider, for example, a single qubit in a magnetic field of known direction but unknown magnitude, so that

$$H_\omega = \frac{\omega}{2} \sigma_3, \quad (25)$$

and let us imagine that the value of the frequency ω is chosen equiprobably from among $N = 2^n$ equally spaced possible values. Without loss of generality, we may normalize the field so that the possible values range from 0 to $1 - 2^{-n}$; then ω has a binary expansion

$$\omega = .\omega_1\omega_2\dots\omega_n \quad (26)$$

that terminates after at most n bits.

The initial state $|\psi_0\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$ evolves in time t to

$$|\psi(t)\rangle_\omega = \exp(-itH_\omega)|\psi_0\rangle = 2^{-1/2}(|0\rangle + \exp(-i\omega t)|1\rangle) \quad (27)$$

(up to an overall phase). If we wait for a time $t_n = \pi 2^n$, the final state is

$$|\psi(t_n)\rangle_\omega = 2^{-1/2}(|0\rangle + \exp(-i\pi\omega_n)|1\rangle). \quad (28)$$

Now measurement in the $\{2^{-1/2}(|0\rangle \pm |1\rangle)\}$ basis indicates (with certainty) whether the bit ω_n is 0 or 1. This outcome divides the set of possible Hamiltonians in half, providing one bit of classical information.

The set of remaining possible Hamiltonians is still evenly spaced, but it may have a constant offset, depending on the value of ω_n . However, the value of ω_n is now known, so the offset can be eliminated. Specifically, if we again prepare $|\psi_0\rangle$ and now evolve for a time $t_{n-1} = \pi 2^{n-1}$, we obtain the final state

† Actually, in a Hilbert space of high dimension, we can make a more complete measurement that will typically return the result that neither H_1 nor H_2 is the actual Hamiltonian.

$$|\psi(t_{n-1})\rangle_\omega = 2^{-1/2}(|0\rangle + \exp[-i\pi(\omega_{n-1}\omega_n)]|1\rangle). \quad (29)$$

Since ω_n is known, we can perform a phase transformation (perhaps by applying an additional driving magnetic field) to eliminate the phase $\exp[-i\pi(\omega_n)]$; Measuring again in the $\{2^{-1/2}(|0\rangle \pm |1\rangle)\}$ basis determines the value of ω_{n-1} .

By continuing this procedure until all bits of ω are known, we perfectly distinguish the 2^n possible Hamiltonians in just n measurements. The procedure is optimal in the sense that we gain one full bit of information about the Hamiltonian in each measurement.

Up until now we have imagined that the frequency ω takes one of 2^n equally spaced discrete values, but no such restriction is really necessary. Indeed, what we have described is precisely the implementation of the n -qubit semi-classical quantum Fourier transform as formulated by Griffiths and Niu [28] (whose relevance to phase estimation was emphasized by Cleve *et al.* [29]). Thus the same procedure can be applied to obtain an estimate of the frequency to n -bit precision, even if the frequency is permitted to take an arbitrary real value in the interval $[0, 1)$.

Suppose that we attach to n spins the labels $\{0, 1, \dots, n-2, n-1\}$, and expose the k th spin to the field for time $\pi 2^{k+1}$; we thus prepare the n -qubit state

$$\prod_{k=0}^{n-1} 2^{-1/2}(|0\rangle + \exp(-i\pi\omega 2^{k+1})|1\rangle) = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} \exp(-2\pi i\omega y)|y\rangle. \quad (30)$$

The adaptive algorithm is equivalent to the quantum Fourier transform followed by measurement; hence the n -bit measurement outcome $\tilde{\omega}$ occurs with probability

$$\text{Prob}_\omega(\tilde{\omega}) = \left| \frac{1}{2^n} \sum_{y=0}^{2^n-1} \exp[-2\pi i y(\omega - \tilde{\omega})] \right|^2. \quad (31)$$

If ω really does terminate in n bits, then the outcome $\tilde{\omega}$ is guaranteed to be its correct binary expansion. But even if the binary expansion of ω does not terminate, the probability that our estimate $\tilde{\omega}$ is correct to n bits of precision is still of order one†.

Of course, to measure the frequency to a precision $\Delta\omega$ of order 2^{-n} , we need to expose our probe spins to the unknown Hamiltonian for a total time T of order $2\pi 2^n$. The accuracy is limited by an energy-time uncertainty relation of the form $T\Delta\omega \sim 1$.

The semi-classical quantum Fourier transform provides an elegant solution to the problem of performing an ideal ‘phase measurement’ in the Hilbert space of n qubits. More broadly, any N -dimensional Hilbert space with a preferred basis $\{|k\rangle, k = 0, 1, \dots, N-1\}$ has a complementary basis of *phase states*

$$|\varphi\rangle = \frac{1}{N^{1/2}} \sum_{k=0}^{N-1} \exp(ik\varphi)|k\rangle, \quad (32)$$

with

$$\varphi = 2\pi j/N, \quad j = 0, 1, \dots, N-1. \quad (33)$$

† We might also use the quantum Fourier transform to *compute* eigenvalues of a known many-body Hamiltonian, rather than *measure* eigenvalues of an unknown one [30].

For example, the Hilbert space could be the truncated space of a harmonic oscillator like a mode of the electromagnetic field, with the occupation number restricted to be less than N ; then the states $|\varphi\rangle$ are the ‘phase squeezed’ states of the oscillator that have minimal phase uncertainty. Since a POVM in an N -dimensional Hilbert space can acquire no more than $\log N$ bits of information about the preparation of the quantum state, the phase of an oscillator with occupation number less than N can be measured to at best $\log N$ bits of accuracy. While it is easy to do an orthogonal measurement in the occupation number basis with an efficient photodetector, an orthogonal measurement in the $|\varphi\rangle$ basis is quite difficult to realize in the laboratory [31].

But if the standard basis is the computational basis in the 2^n -dimensional Hilbert space of n qubits, then an ideal phase measurement is simple to realize. Since the phase eigenstates are actually not entangled states, we can carry out the measurement—*adaptively*—one qubit at a time.

Note that if we had an arbitrarily powerful quantum computer with an arbitrarily large amount of quantum memory, then adaptive measurement strategies might seem superfluous. We could achieve the same effect by introducing a large ancilla and a driving Hamiltonian that acts on probe and ancilla, with all measurements postponed to the very end. But the semi-classical quantum Fourier transform illustrates that adaptive techniques can reduce the complexity of the quantum information processing required to perform the measurement. In many cases, an adaptive strategy may be realizable in practice, while the equivalent unitary strategy is completely infeasible.

8. Distinguishability and decoherence

In all of our examples so far, we have ignored noise and decoherence. In practice, decoherence may compromise our ability to decipher the classical signal with high confidence. Finding ways to improve measurement accuracy by effectively coping with decoherence is an important challenge faced by quantum information theory.

If there is decoherence, our aim is to gain information about the value of a parameter in a master equation rather than a Hamiltonian. To be concrete, consider a single qubit governed by an unknown Hamiltonian H , and also subject to decoherence described by the ‘depolarizing channel’; the density matrix ρ of the qubit obeys the master equation

$$\dot{\rho} = -i[H, \rho] - \Gamma(\rho - \frac{1}{2}I), \quad (34)$$

where Γ is the (known) damping rate. If we express ρ in terms of the polarization vector \mathbf{P} ,

$$\rho = \frac{1}{2}(I + \mathbf{P} \cdot \boldsymbol{\sigma}), \quad (35)$$

and the Hamiltonian as

$$H = \frac{\omega}{2} \hat{a} \cdot \boldsymbol{\sigma}, \quad (36)$$

then the master equation becomes

$$\dot{\mathbf{P}} = \omega(\hat{a} \times \mathbf{P}) - \Gamma\mathbf{P}. \quad (37)$$

The polarization precesses uniformly with circular frequency ω about the \hat{a} axis as it contracts with lifetime Γ^{-1} .

Suppose that we are to distinguish among two possible Hamiltonians, which are assumed to be equiprobable. If we are able to add a driving term, we may assume that the two are the trivial Hamiltonian and

$$H = \frac{\omega}{2} \sigma_3. \quad (38)$$

We choose the initial polarization vector $P_0 = (1, 0, 0)$. Then if the Hamiltonian is trivial, the polarization contracts as

$$\mathbf{P}(t)_{\text{triv}} = \exp(-\Gamma t)(1, 0, 0), \quad (39)$$

while under the non-trivial Hamiltonian it contracts and rotates as

$$\mathbf{P}(t)_{\text{nontriv}} = \exp(-\Gamma t)(\cos \omega t, \sin \omega t, 0). \quad (40)$$

When is the best time to measure the polarization? We should wait until \mathbf{P}_{triv} and $\mathbf{P}_{\text{nontriv}}$ point in distinguishable directions, but if we wait too long, the states will depolarize. The optimal measurement to distinguish the two is an orthogonal measurement of the polarization along the axis normal to the bisector of the vectors $\mathbf{P}(t)_{\text{triv}}$ and $\mathbf{P}(t)_{\text{nontriv}}$. At time t the probability that this measurement identifies the Hamiltonian incorrectly is

$$P_{\text{error}} = \frac{1}{2} - \frac{1}{2} \exp(-\Gamma t) \left| \sin\left(\frac{\omega t}{2}\right) \right|. \quad (41)$$

This error probability is minimized, and the information gain from the measurement is maximized, at a time t such that

$$\tan\left(\frac{\omega t}{2}\right) = \frac{\omega}{2\Gamma}. \quad (42)$$

If $\Gamma/\omega \ll 1$, this time is close to π/ω , the time we would measure to perfectly distinguish the Hamiltonians in the absence of decoherence. But if $\Gamma/\omega \gg 1$, then we should measure after a time $t \sim \Gamma^{-1}$ comparable to the lifetime.

More generally, consider an ensemble of two density operators ρ_1 and ρ_2 with *a priori* probabilities p_1 and p_2 (where $p_1 + p_2 = 1$), and imagine that an unknown state has been drawn from this ensemble. A procedure for deciding whether the unknown state is ρ_1 or ρ_2 can be modelled as a POVM with two outcomes. The two-outcome POVM that minimizes the probability of making an incorrect decision is a measurement of the orthogonal projection onto the space spanned by the eigenstates of $p_1\rho_1 - p_2\rho_2$ with positive eigenvalues [6, 32]. The minimal error probability achieved by this measurement is

$$P_{\text{error}} = \frac{1}{2} - \frac{1}{2} \text{tr} |p_1\rho_1 - p_2\rho_2|. \quad (43)$$

Correspondingly, if we are to identify an unknown superoperator as one of \mathbb{S}_1 and \mathbb{S}_2 (with *a priori* probabilities p_1 and p_2), then the way to distinguish $\mathbb{S}_1, \mathbb{S}_2$ with minimal probability of error is to choose our initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ to minimize†

$$P_{\text{error}} = \frac{1}{2} - \frac{1}{2} \text{tr} |(p_1\mathbb{S}_1 - p_2\mathbb{S}_2)\rho_0|. \quad (44)$$

† We thank Chris Fuchs for a helpful discussion of this point.

In the case of interest to us, the superoperators \mathcal{S}_1 and \mathcal{S}_2 are obtained by integrating, for time t , master equations with Hamiltonians H_1 and H_2 respectively. We minimize the error probability in equation (44) with respect to t to complete the optimization.

9. Entanglement and frequency measurement

Consider again the case in which the Hamiltonian is known to be of the form

$$H_\omega = \frac{\omega}{2} \sigma_3, \quad (45)$$

but where the frequency ω is unknown. For the moment, let us neglect decoherence, but suppose that we have been provided with a large number n of qubits that we may use to perform an experiment to determine ω in a *fixed total time* t . What is the most effective way to employ our qubits?

Consider two strategies. In the first, we prepare n identical qubits polarized along the x axis. They precess in the field described by H_ω for time t , and then the spin along the x axis is measured. Each spin will be found to be pointing 'up' with probability

$$P = \frac{1}{2}(1 + \cos \omega t). \quad (46)$$

Because the measurement is repeated many times, we will be able to estimate the probability P to an accuracy

$$\Delta P = [P(1 - P)/n]^{1/2} = \frac{|\sin \omega t|}{2n^{1/2}}. \quad (47)$$

and so determine the value of ω to accuracy

$$\Delta \omega = \frac{\Delta P}{t|dP/d(\omega t)|} = \frac{1}{tn^{1/2}}. \quad (48)$$

The accuracy improves like $1/n^{1/2}$ as we increase the number of available qubits with the time t fixed.

The second strategy is to prepare an entangled 'cat' state of n ions

$$|\psi_0\rangle = 2^{-1/2}(|000\dots 0\rangle + |111\dots 1\rangle). \quad (49)$$

The advantage of the entangled state is that it precesses n times faster than a single qubit; in time t it evolves to

$$|\psi(t)\rangle = 2^{-1/2}(|000\dots 0\rangle + \exp(in\omega t)|111\dots 1\rangle) \quad (50)$$

(up to an overall phase). If we now perform an orthogonal measurement that projects onto the basis $2^{-1/2}(|000\dots 0\rangle \pm |111\dots 1\rangle)$ (e.g. a measurement of the entangled observable $\sigma_1 \otimes \sigma_1 \otimes \dots \otimes \sigma_1$) then we will obtain the '+' outcome with probability

$$P = \frac{1}{2}(1 + \cos n\omega t). \quad (51)$$

By this method, $n\omega t$ can be measured to order one accuracy, so that

$$\Delta \omega \simeq \frac{1}{tn}, \quad (52)$$

a more favourable scaling with n than in equation (48).

This idea of exploiting the rapid precession of entangled states to achieve a precision beyond the shot-noise limit has been proposed in both frequency measurement [33] and optical interferometry [34]. (One realization of this idea is the proposal by Caves [35] to allow a squeezed vacuum state to enter the dark port of an interferometer; the squeezing induces the n photons entering the other port to make correlated 'decisions' about which arm of the interferometer to follow.)

10. Entanglement versus decoherence

In both equation (48) and equation (52), the accuracy of the frequency measurement improves with the elapsed time t as $1/t$. But so far we have neglected decoherence. If the single-qubit state decays at a rate Γ , then we have seen that the optimal time at which to perform a measurement will be of order Γ^{-1} . The entangled strategy will still be better if we are constrained to perform the measurement in a time $t \ll \Gamma^{-1}$, but further analysis is needed to determine which method is better if we are free to choose the time t to optimize the accuracy.

In fact, as Huelga *et al.* [16] have emphasized, an entangled state is fragile, and its faster precession can be offset by its faster decay rate. Suppose that two qubits are available, both independently subjected to the depolarizing channel with decay rate Γ . If we prepare the unentangled state, each qubit has the initial pure-state density matrix

$$\rho_0 = \frac{1}{2}(I + \sigma_1) \quad (53)$$

polarized along the x axis, and evolves in time t to

$$\rho(t) = \frac{1}{2}[I + \exp(-\Gamma t)(\sigma_1 \cos \omega t + \sigma_2 \sin \omega t)]. \quad (54)$$

If we now measure σ_1 , we obtain the $+$ result with probability

$$P = \text{tr}\left(\frac{1}{2}(I + \sigma_1)\rho(t)\right) = \frac{1}{2}(1 + \exp(-\Gamma t) \cos \omega t). \quad (55)$$

Now suppose that the initial state is the Bell state $|\phi^+\rangle$ of two qubits, with density matrix

$$\rho_0 = \frac{1}{4}(I \otimes I + \sigma_3 \otimes \sigma_3 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2). \quad (56)$$

If both spins precess and depolarize independently, this state evolves to

$$\begin{aligned} \rho(t) = & \frac{1}{4}[I \otimes I + \exp(-2\Gamma t)(\sigma_3 \otimes \sigma_3 \\ & + \cos 2\omega t(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) \\ & + \sin 2\omega t(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1))]; \end{aligned} \quad (57)$$

if we measure the observable $\sigma_1 \otimes \sigma_1$, we find the $+$ outcome with probability

$$\begin{aligned} P = & \text{tr}\left(\frac{1}{2}(I \otimes I + \sigma_1 \otimes \sigma_1)\rho(t)\right) \\ = & \frac{1}{2}(1 + \exp(-2\Gamma t) \cos 2\omega t). \end{aligned} \quad (58)$$

Note that equation (58) has exactly the same functional form as equation (55), but with t replaced by $2t$. Therefore, the entangled measurement performed in time $t/2$ collects exactly as much information about the frequency ω as the measurement of a single ion performed in time t . If we have two qubits and

total time t available, we can either perform the entangled measurement twice (taking time $t/2$ each time), or perform measurements on each qubit independently (taking time t). Either way, we obtain two outcomes and collect exactly the same amount of information on average.

More generally, suppose that we have n qubits and a total time $T \gg 1/\Gamma$ available. We can use these qubits to perform altogether nT/t independent single-qubit measurements, where each measurement requires time t . Plugging equation (55) into equation (47) and equation (48) (with n replaced by nT/t), and choosing $\cos \omega t \sim 0$ to optimize the precision, we find that the frequency can be determined to accuracy

$$\Delta\omega = \left(\frac{1}{t}\right) \frac{\exp(\Gamma t)}{(nT/t)^{1/2}} = \frac{1}{(nT)^{1/2}} \frac{\exp(\Gamma t)}{t^{1/2}}. \quad (59)$$

This precision is optimized if we choose $\Gamma t = 1/2$, where we obtain [16]

$$\Delta\omega = \left(\frac{2e\Gamma}{nT}\right)^{1/2}. \quad (60)$$

On the other hand, we could repeat the experiment T/t times using the n -qubit entangled state. Then we would obtain a precision

$$\Delta\omega = \left(\frac{1}{nt}\right) \frac{\exp(n\Gamma)t}{(T/t)^{1/2}} = \frac{1}{(nT)^{1/2}} \frac{\exp(n\Gamma t)}{(nt)^{1/2}}, \quad (61)$$

the same function as for uncorrelated qubits, but with t replaced by nt . Thus the optimal precision is the same in both cases, but is attained in the uncorrelated case by performing experiments that take n times longer than in the correlated case.

That the entangled states offer no advantage in the determination of ω was one of the main conclusions of Huelga *et al.* [16]. A similar conclusion applies to estimating the difference in path length between two arms of an interferometer using a specified optical power, if we take into account losses and optimize with respect to the number of times the light bounces inside the interferometer before it escapes and is detected.

We would like to make the (rather obvious) point that this conclusion can change if we adopt a different model of decoherence, and in particular if the qubits do not decohere independently. As a simple example of correlated decoherence, consider the case of two qubits with 4×4 density matrix ρ evolving according to the master equation

$$\dot{\rho} = -i[H, \rho] - \Gamma(\rho - I/4). \quad (62)$$

This master equation exhibits the analogue, in the four-dimensional Hilbert space, of the uniform contraction of the Bloch sphere described by the depolarizing channel in the case of a qubit. Because the decoherence picks out no preferred direction in the Hilbert space (or any preferred tensor-product decomposition), we call this model 'symmetric decoherence'.

Under this master equation, with both qubits subjected to H_ω and to symmetric decoherence, the Bell state $\rho_0 = |\phi^+\rangle\langle\phi^+|$ evolves in time t to the state

$$\begin{aligned} \rho(t) = & \frac{1}{4} [I \otimes I + \exp(-\Gamma t)(\sigma_3 \otimes \sigma_3 \\ & + \cos 2\omega t(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) \\ & + \sin 2\omega t(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1))], \end{aligned} \quad (63)$$

so that a measurement of $\sigma_1 \otimes \sigma_1$ yields the + outcome with probability

$$P = \frac{1}{2}(1 + \exp(-\Gamma t) \cos 2\omega t). \quad (64)$$

On the other hand, the initial product state

$$\rho_0 = \frac{1}{4}(I + \sigma_1) \otimes (I + \sigma_1) \quad (65)$$

becomes entangled as a result of symmetric decoherence. If the Hamiltonian were trivial, it would evolve to

$$\rho(t) = \frac{1}{4} I \otimes I + \frac{1}{4} \exp(-\Gamma t)(\sigma_1 \otimes I + I \otimes \sigma_1 + \sigma_1 \otimes \sigma_1). \quad (66)$$

Including the precession

$$\sigma_1 \rightarrow \sigma_1 \cos \omega t + \sigma_2 \sin \omega t, \quad (67)$$

we obtain

$$\rho(t) = \frac{1}{4} I \otimes I + \frac{1}{4} \exp(-\Gamma t)(\sigma_1 \otimes I \cos \omega t + \dots), \quad (68)$$

so that measurement of the single-qubit observable $\sigma_1 \otimes I$ yields the + outcome with probability

$$P = \text{tr} \left(\frac{1}{2}(I \otimes I + \sigma_1 \otimes I) \rho(t) \right) = \frac{1}{2}(1 + \exp(-\Gamma t) \cos \omega t). \quad (69)$$

Comparing equation (69) and equation (64), the important thing to notice is that with symmetric decoherence, entangled states decay no faster than product states; therefore, we can enjoy the benefit of entanglement (faster precession) without paying the price (faster decay).

To establish more firmly that entangled strategies outperform non-entangled strategies in the symmetric decoherence model, we should consider more closely what are the optimal final measurements for these two types of initial states. To give the problem a precise information-theoretic formulation, we return to the problem of distinguishing two cases, the trivial Hamiltonian and H_ω , which are assumed to be equiprobable. For either the product initial state or the entangled initial state, we evolve for time t , and then perform the best measurement that distinguishes between evolution governed by H_ω and trivial evolution. In both cases, the measurement is permitted to be an entangled measurement; that is, we optimize with respect to all POVMs in the four-dimensional Hilbert space.

In either case (initial product state or initial entangled state), we can find the two-outcome POVM that identifies the Hamiltonian with minimal probability of error. When there is no decoherence, this POVM (when restricted to the two-dimensional subspace containing the two pure states to be distinguished) is the familiar orthogonal measurement that best distinguishes two pure states of a qubit. In fact, for symmetric decoherence, this same measurement minimizes the error probability for any value of the damping rate Γ . It is also the two-outcome measurement with the maximal information gain (the measurement outcome has maximal mutual information with the choice of the Hamiltonian). Although we do not have a proof, we can make a reasonable guess that, for symmetric decoherence,

this two-outcome measurement has the maximal information gain of any measurement, including POVMs with more outcomes.

If either initial state evolves for time t , and then this optimal POVM is performed, the error probability can be expressed as

$$P_{\text{error}} = \frac{1}{2} - \frac{1}{2} \exp(-\Gamma t) |\sin \theta(t)|; \tag{70}$$

here $\theta(t)$ is the angle between the states—that is, $\cos \theta(t)$ is the inner product of the evolving and static states, in the limit of no damping ($\Gamma = 0$). For the entangled initial state, we have

$$\theta_{\text{entangled}} = \omega t, \tag{71}$$

and for the product initial state, we have

$$\cos \theta_{\text{product}} = \cos^2 \left(\frac{\omega t}{2} \right). \tag{72}$$

Since

$$|\cos \theta_{\text{entangled}}| = |\cos \omega t| \leq \frac{1}{2}(1 + \cos \omega t) = |\cos \theta_{\text{product}}| \tag{78}$$

for $\cos \theta_{\text{entangled}} \geq 0$, the error probability achieved by the entangled initial state is smaller than that achieved by the product state for $0 < \omega t < \pi/2$, which is sufficient to ensure that the error probability optimized with respect to t is always smaller in the entangled case for any non-zero value of Γ . Similarly, if we optimize the information gain with respect to t , the entangled strategy has the higher information gain for all $\Gamma > 0$. The improvement in information gain (in bits) achieved using an entangled initial state rather than a product initial state is plotted in figure 1 as a function of Γ/ω . The maximum improvement of about 0.136 bits occurs for $\Gamma/\omega \sim 0.404$.

We have already seen in section 2 that, even in the absence of decoherence, an entangled strategy may outperform an unentangled strategy if we are trying to distinguish more than two alternatives. This advantage will persist when suffi-

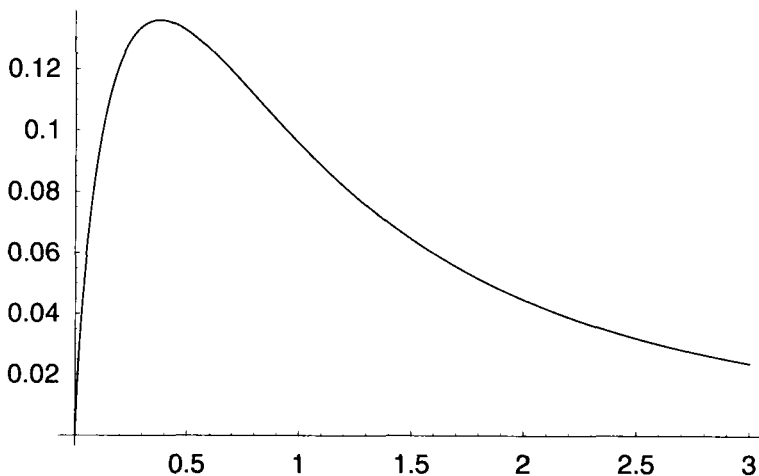


Figure 1. Improvement in information gain (in bits) achieved by using an entangled initial state, as a function of the ratio of decoherence rate Γ to precession frequency ω .

ciently weak decoherence is included, whether correlated or uncorrelated. In that event, since only one member of an entangled pair is exposed to the unknown Hamiltonian, we may be able to shelter the other member of the pair from the ravages of the environment, slowing the decay of the state and strengthening the signal.

11. Conclusions

We feel that quantum information theory, having already secured a central position at the foundations of computer science, will eventually erect bridges connecting with many subfields of physics. The results reported here (and other related examples) give strong hints that ideas emerging from the theory of quantum information and computation are destined to profoundly influence the experimental physics techniques of the future.

We have only scratched the surface of this important subject. Among the many issues that deserve further elaboration are the connections between superoperator distinguishability and superoperator norms, the efficacy of the quantum Fourier transform in the presence of decoherence, the measurement of continuous quantum variables, the applications of quantum error correction, and the detection of time-dependent signals.

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Appendix A: fixed-time-driving theorem

In this appendix, we sketch the proof of the theorem stated in section 5.

For a unitary $N \times N$ matrix U , we define $\max\arg(U)$ to be the largest argument of an eigenvalue of U , where the argument takes values in the interval $(-\pi, \pi]$. Similarly, $\min\arg(U)$ is the minimum argument of an eigenvalue of U . Our theorem is:

Theorem 1: *If H and K are finite-dimensional Hermitian matrices, and $\|H\|_{\text{sup}} < \pi$, then*

$$\text{maxarg}(\exp(iK) \exp[-i(H + K)]) \leq \text{maxarg}(\exp(-iH)), \tag{A 1}$$

$$\text{minarg}(\exp(iK) \exp[-i(H + K)]) \geq \text{minarg}(\exp(-iH)). \tag{A 2}$$

To prove the theorem, we begin with:

Lemma 1: *For unitary U with $\text{maxarg}(U) \neq \pi$, and Hermitian A ,*

$$\text{maxarg}(U \exp(i\epsilon A)) \leq \text{maxarg}(U) + \text{maxarg}(\exp(i\epsilon A)) + O(\epsilon^2), \tag{A 3}$$

$$\text{minarg}(U \exp(i\epsilon A)) \geq \text{minarg}(U) + \text{minarg}(\exp(i\epsilon A)) - O(\epsilon^2). \tag{A 5}$$

Proof: Write $U = \exp(iB)$, where B is Hermitian and $\|B\|_{\text{sup}} < \pi$; then $\text{maxarg}(\exp(iB)) = \max(B)$, where $\max(B)$ denotes the maximum eigenvalue of B . From the Baker–Campbell–Hausdorff formula, we have

$$\exp(iB) \exp(i\epsilon A) = \exp\left(i\left(B + \epsilon A + \frac{i}{2}\epsilon[C, B] + O(\epsilon^2)\right)\right), \tag{A 7}$$

where C is linear in A . Then lowest-order eigenvalue perturbation theory tells us that

$$\begin{aligned} \max\left(B + \epsilon A + \frac{i}{2}\epsilon[C, B]\right) &= \max(B) + \langle \psi | \left(\epsilon A + \frac{i}{2}\epsilon[C, B]\right) | \psi \rangle + O(\epsilon^2) \\ &= \max(B) + \langle \psi | (\epsilon A) | \psi \rangle + O(\epsilon^2) \\ &\leq \max(B) + \max(\epsilon A) + O(\epsilon^2) \end{aligned} \tag{A 11}$$

(where $|\psi\rangle$ is in the eigenspace of B with maximal eigenvalue). This proves equation (A3). Equation (A4) is proved similarly. Note that the condition $\text{maxarg}(U) \neq \pi$ is necessary so that the singularity of the maxarg function can be avoided for ϵ sufficiently small.

Lemma 1 is all we will need for the proof of Theorem 1. But it is useful to note that Lemma 1 may be invoked to prove:

Lemma 2†: *For unitary U_1 and U_2 , such that*

$$\text{maxarg}(U_1) + \text{maxarg}(U_2) < \pi, \tag{A 7}$$

$$\text{minarg}(U_1) + \text{minarg}(U_2) > -\pi, \tag{A 8}$$

we have

$$\text{maxarg}(U_1 U_2) \leq \text{maxarg}(U_1) + \text{maxarg}(U_2), \tag{A 9}$$

$$\text{minarg}(U_1 U_2) \geq \text{minarg}(U_1) + \text{minarg}(U_2). \tag{A 10}$$

Proof: We write

† This is actually a special case of a result proved in [36]. A recent proof appears in [37].

$$U_1 U_2 = U_1 \exp(iA) = U_1 (\exp(iA/n))^n, \quad (\text{A } 11)$$

where the eigenvalues of A lie in the interval $(-\pi, \pi)$, and apply Lemma 1 repeatedly, obtaining

$$\begin{aligned} \maxarg(U_1 \exp(iA)) &\leq \maxarg(U_1) \\ &\quad + n[\maxarg(\exp(iA/n)) + O(n^{-2})]. \end{aligned} \quad (\text{A } 12)$$

Taking the $n \rightarrow \infty$ limit proves equation (A 9). Equation (A 10) is proved similarly. Note that because of the conditions equation (A 7) and equation (A 8), Lemma 1 can be safely applied n times in succession; the accumulated \maxarg and \minarg of the product never approach π .

To complete the proof of Theorem 1, we invoke the Lie product formula

$$\lim_{n \rightarrow \infty} (\exp(A/n) \exp(B/n))^n = \exp(A + B), \quad (\text{A } 13)$$

to write

$$\begin{aligned} \exp(iK) \exp[-i(H + K)] &= \lim_{n \rightarrow \infty} (\exp(iK/n))^n (\exp(-iH/n) \exp(-iK/n))^n \\ &= \lim_{n \rightarrow \infty} \exp(iK/n) \dots \exp(iK/n) \\ &\quad \exp(-iH/n) \exp(-iK/n) \dots \exp(-iH/n) \exp(-iK/n). \end{aligned} \quad (\text{A } 14)$$

Since $\exp(iK/n) \exp(-iH/n) \exp(-iK/n)$ and $\exp(-iH/n)$ have the same eigenvalues, Lemma 2 implies that

$$\begin{aligned} \maxarg(\exp(iK/n) \exp(-iH/n) \exp(-iK/n) \exp(-iH/n)) \\ \leq 2 \maxarg(\exp(-iH/n)). \end{aligned} \quad (\text{A } 15)$$

Similarly, we have

$$\begin{aligned} \maxarg(\exp(iK/n) \exp(iK/n) \exp(-iH/n) \exp(-iK/n) \exp(-iH/n)) \\ \exp(-iK/n) \exp(-iH/n) \leq 3 \maxarg(\exp(-iH/n)), \end{aligned} \quad (\text{A } 16)$$

and so on. Hence, applying Lemma 2 altogether n times to the right-hand side of equation (A 14), we find that

$$\begin{aligned} \maxarg(\exp(iK) (\exp(-iH/n) \exp(-iK/n))^n) &\leq n \maxarg(\exp(-iH/n)) \\ &= \maxarg(\exp(-iH)). \end{aligned} \quad (\text{A } 17)$$

Taking the $n \rightarrow \infty$ limit completes the proof of equation (A 1). Equation (A 2) is proved similarly.

The upper bound on $\|H\|_{\text{sup}}$ is a key feature of the formulation of Theorem 1. This bound ensures that the conditions equation (A 7) and equation (A 8) are satisfied each time that Lemma 2 is invoked in the proof. If $\|H\|_{\text{sup}}$ is too large, then counterexamples can be constructed.

In any event, for the discussion in section 5, we are interested in the case where the maximal and minimal eigenvalues of H differ by less than π , and by shifting H by a constant we can ensure that $\|H\|_{\text{sup}} < \pi/2$. Therefore, the theorem enforces the conclusion that if we are to distinguish a non-trivial Hamiltonian from the

trivial Hamiltonian in an experiment conducted in a fixed elapsed time, turning on a non-zero time-independent 'driving term' K provides no advantage.

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