Non-Abelian vortices and non-Abelian statistics

Hoi-Kwong Lo and John Preskill
California Institute of Technology, Pasadena, California 91125
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We study the interactions of non-Abelian vortices in two spatial dimensions. These interactions have novel features, because the Aharonov-Bohm effect enables a pair of vortices to exchange quantum numbers. The cross section for vortex-vortex scattering is typically a multivalued function of the scattering angle. There can be an exchange contribution to the vortex-vortex scattering amplitude that adds coherently with the direct amplitude, even if the two vortices have distinct quantum numbers. Thus two vortices can be "indistinguishable" even though they are not the same.

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I. INTRODUCTION

It is well known that exotic generalizations of fermion and boson statistics are possible in two spatial dimensions. The simplest, and most familiar, such generalization is anyon statistics [1, 2]. When two indistinguishable anyons are adiabatically interchanged (or one anyon is rotated by 2π), the many-body wave function acquires the phase $e^{i\theta}$, where θ can take any value. An instructive example of an object that obeys anyon statistics is a composite of a magnetic vortex (with magnetic flux Φ) and a charged particle (with charge q) [2]. Then the anyon phase arises as a consequence of the Aharonov-Bohm effect, with $e^{i\theta} = e^{iq\Phi}$. Furthermore, anyon statistics is actually known to be realized in nature, in systems that exhibit the fractional quantum Hall effect [3].

It is natural to consider a further generalization: non-Abelian statistics [4–7]. A particular type of non-Abelian statistics is realized by the non-Abelian vortices (and vortex-charge composites) that occur in some spontaneously broken gauge theories. Loosely speaking, the unusual feature of the many-body physics in this case is that the quantum numbers of an object depend on its *history*. In particular, if one vortex is adiabatically carried around another, the quantum numbers of both may change, due to a non-Abelian variant of the Aharonov-Bohm effect. Thus, whether two bodies are identical is not a globally defined notion.

There is no firm observational evidence for the existence of objects that obey this type of quantum statistics. Perhaps such objects will eventually be found in suitable condensed matter systems. (Analogous non-Abelian defects associated with spontaneous breakdown of *global* symmetries are observed in liquid crystals [8] and ^{3}He [9].) In any event, the physics of non-Abelian vortices is intrinsically interesting and instructive. For one thing, it forces us to carefully consider some subtle aspects of non-Abelian gauge invariance.

In this paper, we will focus on the Aharonov-Bohm interactions of a pair of non-Abelian vortices. This is, of course, much simpler and much less interesting than the problem of three or more bodies. Nevertheless, an important conceptual point will be illuminated by our calculation of the vortex-vortex scattering cross section. We will see that this cross section is in general multivalued. While we have learned to be undisturbed, at least in certain contexts, by multivalued wave functions, a cross section is directly observable and so is ordinarily expected to be a single-valued function of the scattering angle. But the multivaluedness of the cross section for vortex-vortex scattering follows naturally from the ambiguity in assigning quantum numbers to the vortices.

Indeed, multivalued scattering cross sections are a generic consequence of the non-Abelian Aharonov-Bohm effect—they arise in the scattering of a charge off a vortex as well. It is useful to consider the case of the "Alice" vortex [10–13], which has the property that when a positively charged particle is adiabatically transported around the vortex, it becomes negatively charged. When a positively charged particle scatters from an Alice vortex, the scattered particle may be either positively charged or negatively charged. Thus there are two measurable exclusive cross sections[1] $\sigma_+(\theta)$ and $\sigma_-(\theta)$. Though the inclusive cross section $\sigma_{\rm inc} = \sigma_+ + \sigma_-$ is single valued, the exclusive cross sections are not; they are double valued and obey the conditions

$$\sigma_+(\theta + 2\pi) = \sigma_-(\theta) , \qquad \sigma_-(\theta + 2\pi) = \sigma_+(\theta) . \tag{1}$$

The double valuedness of the exclusive cross sections is an unavoidable consequence of the feature that a charged particle that voyages around the Alice vortex returns to its starting point with its charge flipped in sign. We might imagine measuring the θ dependence of the cross section by gradually transporting a particle detector around the scattering center. But then a detector that has been designed to respond to positively charged particles will have become a detector that responds to negatively charged particles when it returns to its starting point. Alternatively, we might catch the scattered particle, and then carry it back along a specified path to a central laboratory for analysis. But then the outcome of the analysis will depend upon the path taken.

[1] Strictly speaking, these cross sections do not exist, because there are no asymptotic charged particle states in two-dimensional electrodynamics; see Sec. VII for further discussion.

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While we may (arbitrarily) associate a definite path with each value of the scattering angle, this path cannot vary continuously with θ. A convention for choosing the path artificially restricts the exclusive cross sections to a single branch of the two-valued function, and introduces a discontinuity in the measured cross sections. As we will discuss in more detail below, the cross sections for non-Abelian vortex-vortex scattering have similar multivaluedness properties.

In the case of vortex-vortex scattering (unlike the case of scattering a charged particle off of a vortex), effects of quantum statistics can be exhibited. That is, there may be an exchange contribution to the scattering amplitude that interferes with the direct amplitude. The existence of an exchange contribution means that the two vortices must be regarded as indistinguishable particles—it is not possible in principle to keep track of which vortex is which. The unusual feature of non-Abelian vortex-vortex scattering is that exchange scattering can occur even if the initial vortices are objects with distinct quantum numbers. The vortices are different, yet they are indistinguishable.

Much that we will say in this paper has been anticipated elsewhere. That the quantum numbers of non-Abelian vortices cannot be globally defined was first emphasized by Bais [14]. Wilczek and Wu [6] and Bucher [7] discussed the implications for vortex-vortex scattering. Verlinde [15] worked out a general formula for the *inclusive* cross section in Aharonov-Bohm scattering, in terms of the matrix elements of the "monodromy" operator, and Bais *et al.* [16] developed a powerful algebraic machinery that can be used to compute these matrix elements (among other things). The main new contributions here are a computation of the *exclusive* cross sections for the various possible quantum numbers of the final state vortices, and an analysis of vortex-vortex scattering that incorporates the exchange of "indistinguishable" vortices. (Wilczek and Wu [6] attempted to calculate the exclusive cross sections, but because they missed the multivaluedness properties of these cross sections, they did not obtain the correct answer.) Once properly formulated, the calculation of these exclusive cross sections is very closely related to the analysis of scattering in (2+1)-dimensional gravity, which was first worked out by 't Hooft [17] and Deser and Jackiw [18].

The remainder of this paper is organized as follows. In Sec. II, we review how the quantum numbers of non-Abelian vortices are modified by an exchange, and we extend the discussion in Sec. III to the case of vortices that also carry charge. We recall the general theory of the quantum mechanics of indistinguishable particles in Sec. IV, and describe how the special case of non-Abelian vortices fits into this general theory. In Sec. V, we calculate the exclusive cross sections for non-Abelian Aharonov-Bohm scattering of a projectile off of a fixed target. The case of vortex-vortex scattering is analyzed in detail, and we emphasize and explain the multivaluedness properties of these cross sections. The case of two-body scattering in the center-of-mass frame is discussed in Sec. VI. This calculation includes the contribution due to the exchange of "indistinguishable" vortices. In Sec. VII, we extend the previous discussion to the case where the unbroken gauge group is continuous, such as the case of the "Alice" vortex. Section VIII contains our conclusions.

II. NON-ABELIAN FLUX AND THE BRAID OPERATOR

We consider, in two spatial dimensions, a gauge theory with underlying gauge group G, which we may take to be connected and simply connected. Suppose that the gauge symmetry is spontaneously broken, and that the surviving manifest gauge symmetry is H. We will assume for now that H is discrete and non-Abelian. The case of continuous H will be briefly discussed in Sec. VII.

This pattern of symmetry breaking will admit stable classical vortex solutions. A vortex carries a "flux" that can be labeled by an element of the unbroken group H. To assign a group element to a vortex, we arbitrarily choose a "base point" x_0 and a path C, beginning and ending at x_0, that winds around the vortex. The effect of parallel transport in the gauge potential of the vortex is then encoded in

$$a(C,x_0) = P\exp\left(i\int_{C,x_0} A\right) \in H(x_0)\ . \tag{2}$$

This group element takes a value in the subgroup $H(x_0)$ of G that preserves the Higgs condensate at the point x_0, since transport of the condensate around the vortex must return it to its original value. If H is discrete, then $a(C,x_0)$ will remain unchanged as the path C is smoothly deformed, as long as the path never crosses the cores of any vortices. (The gauge connection is locally flat outside the vortex cores, with curvature singularities at the cores.)

The flux of a vortex can be measured via the Aharonov-Bohm effect [19, 20]. We can imagine performing a double-slit interference experiment with a beam of particles that transform as some representation R of H. If we then repeat the experiment with the vortex placed between the two slits, the change in the interference pattern reveals

$$\langle u^{(R)}|D^{(R)}(a)|u^{(R)}\rangle\ , \tag{3}$$

where $|u^{(R)}\rangle$ is the internal wave function of the particles in the beam. (The shift in the interference fringes is determined by the phase of this quantity, and the amplitude of the intensity modulation is determined by its modulus.) By measuring this for various $|u^{(R)}\rangle$'s, all matrix elements of $D^{(R)}(a)$ can be determined, and hence, if the representation is faithful, a itself.

However, the flux of the vortex is not a gauge-invariant quantity. A gauge transformation $h \in H(x_0)$ that preserves the Higgs condensate at the base point transforms the flux according to

$$h:\quad a \to hah^{-1}\ . \tag{4}$$

(This gauge transformation is just a relabeling of the particles that are used to perform the measurement of the flux.) Since the gauge transformations act transitively on the conjugacy class in H to which the flux belongs, one might be tempted to say that the flux of a vortex should really be labeled by a conjugacy class rather than a group element. But that is not correct. If there are two vortices, labeled by group elements a and b with respect to the same base point x_0, then the effect of a gauge transformation at x_0 is

$$h: \quad a \to hah^{-1}\ , \quad b \to hbh^{-1}\ . \tag{5}$$

Thus, if a and b are distinct representatives of the same class, they remain distinct in any gauge.

More generally, we can imagine assembling a "vortex bureau of standards," where standard vortices corresponding to each group element are stored. If a vortex of unknown flux is found, we can carry it back to the bureau of standards and determine which of the standard vortices it matches. (Alternatively, we can find out which antivortex it annihilates.) Thus, though there is arbitrariness in how we assign group elements to the standard vortices, once our standards are chosen there is no ambiguity in assigning a label to the new vortex.

We might have said much the same thing about measuring the color of a quark. Although the color is not a gauge-invariant quantity, we can erect a quark bureau of standards in which standard red, yellow, and blue quarks are kept. When a new quark is found, we can carry it back to the bureau and determine its color relative to our standard basis. However, in the case where there are light gauge fields, the curvature of the gauge connection is easily excited. We may find, then, that the outcome of the measurement of the color depends on the path that is chosen when the quark is transported back to the bureau.

In the case where the unbroken gauge group is discrete, there are no light gauge fields. The measurement of the flux of a vortex is unaffected by a deformation of the path that is used to bring the vortex to the bureau of standards, as long as the path does not cross the cores of any other vortices. But when other vortices are present, there is a discrete choice of topologically distinct paths, and the measured flux will in general depend on how we choose to weave the vortex among the other vortices on the way back to the bureau. This ambiguity in measuring the flux is the origin of the "holonomy interaction" among vortices [14] and of Aharonov-Bohm vortex-vortex scattering [6, 7].

To characterize this interaction, we consider how the fluxes assigned to a pair of vortices are modified when the two vortices are adiabatically interchanged, as depicted in Fig. 1. Here α and β are standard paths, beginning and ending at the base point x_0, that are used to define the flux of the two vortices; the corresponding group elements are a and b, respectively. When the two vortices are interchanged (in a counterclockwise sense), these paths can be dragged to new paths α' and β', in such a way that no path ever crosses any vortex. Thus, the group elements associated with transport along α' and β' are, after the interchange, still a and b, respectively. But the final paths are topologically distinct from the initial paths; from Fig. 1(b) we see that

$$\alpha' = \beta \to a\ , \quad \beta' = \beta^{-1}\alpha\beta \to b\ . \tag{6}$$

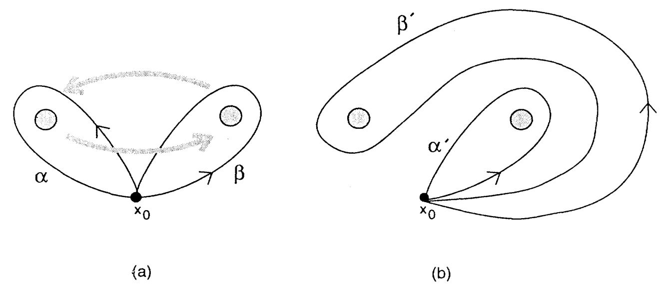

FIG. 1. Exchange of two vortices. (a) The paths α and β are two standard paths, both beginning and ending at the same base point x_0, which are used to define the flux of two vortices. (b) When the vortices are interchanged, these paths are dragged to the new paths $\beta' = \beta^{-1}\alpha\beta$ and $\alpha' = \beta$.

(Here, in order to be consistent with the rules for composing path-ordered exponentials, we have chosen an ordering convention in which $\alpha\beta$ denotes the path obtained by first traversing β, then α.) We conclude that, after the interchange, the effect of parallel transport around α is given by the group element aba^{-1}. The effect of the interchange on the two-vortex state can be expressed as the action of the *braid operator* $\mathcal{R}$, where

$$\mathcal{R}: \quad |a, b\rangle \to |aba^{-1}, a\rangle\ . \tag{7}$$

Naturally, the braid operator preserves the "total flux" ab that is associated with counterclockwise transport around the vortex pair, for this flux could be measured by a particle that is very far away from the pair, and cannot be affected by the interchange. If the interchange is performed twice (which is equivalent to transporting

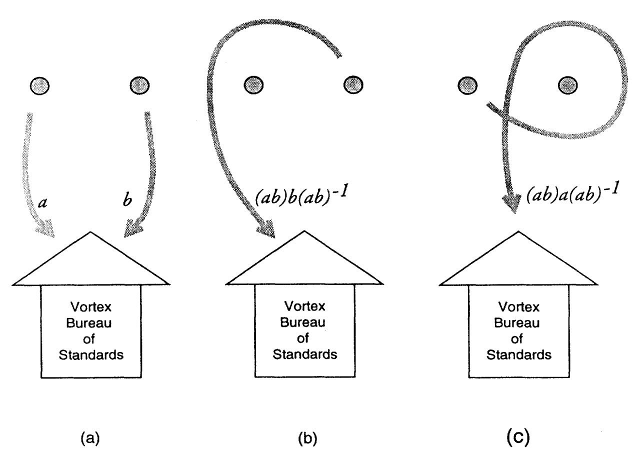

FIG. 2. Vortices can be carried along specified paths to the "vortex bureau of standards," where their flux can be measured. If the two vortices are carried along the paths shown in (a), the fluxes are measured to be a and b, respectively. But if the b vortex goes counterclockwise around the a vortex before voyaging to the bureau, as in (b), its flux is measured as $(ab)b(ab)^{-1}$. If the a vortex goes counterclockwise around the b vortex before voyaging to the bureau, as in (c), its flux is measured as $(ab)a(ab)^{-1}$.

one vortex in a counterclockwise sense about the other), the state transforms according to

$$\mathcal{R}^2: \quad |a,b\rangle \to |(ab)a(ab)^{-1},(ab)b(ab)^{-1}\rangle \; ; \tag{8}$$

both fluxes are conjugated by their combined "total flux" ab.

This result has a clear, gauge-invariant meaning. Suppose that two vortices are carried from their initial positions to the vortex bureau of standards along the paths shown in Fig. 2(a), and are found to have fluxes a and b. Then if they are carried to the bureau along the alternative paths shown in Figs. 2(b,c), the outcome of the flux measurement will be different, as expressed in Eq. (8).

III. FLUX-CHARGE COMPOSITES

The above discussion can be generalized to the case of objects that carry both flux and charge. But there is one noteworthy subtlety. The "charge" of an object is defined by its transformation properties under global gauge transformations. If the object carries flux, however, there is a topological obstruction to implementing the global gauge transformations that do not commute with the flux [21, 11, 12]. If the flux is a, only a subgroup of H, the centralizer $N(a)$ of the flux, is "globally realizable" acting on the vortex. Thus, the charged states of a vortex with flux a transform as a representation of $N(a)$ rather than the full group H.

We can understand the physical meaning of this obstruction if we think about measuring charge via the Aharonov-Bohm effect. The charge can be measured in a double-slit interference experiment, by observing the effect on the interference pattern when various vortices are placed between the slits. But if the particles in the beam carry flux a, and the vortex between the slits carries flux b, then no interference pattern is seen if a and b do not commute. The trouble is that, due to the holonomy interaction, the objects that pass through the respective slits carry different values of the flux when they arrive at the detector, and so do not interfere. (See Fig. 3.) Even more to the point, the slit that the object passed through becomes correlated with the state of the vortex that is placed between the slits, because both fluxes become conjugated as in Eq. (8). Thus, the superposition of particles that passed through the two slits becomes incoherent, and there is no interference. There will be an interference pattern, and a successful charge measurement, only if the flux between the slits commutes with the flux a carried by the particles in the beam. Hence only the transformation properties under $N(a)$ can be measured.

Since the global gauge transformations that can be implemented actually commute with the flux, a non-Abelian vortex that carries charge behaves much like an Abelian flux-charge composite. If the vortex carries flux a and transforms as an irreducible representation $(R^{(a)})$ of $N(a)$, then, since a lies in the center of its centralizer $N(a)$, it is represented by a multiple of the identity in $R^{(a)}$ (by Schur's lemma),

$$D^{(R^{(a)})}(a) = e^{i\theta_{R^{(a)}}}\,\mathbf{1}_{R^{(a)}} \; . \tag{9}$$

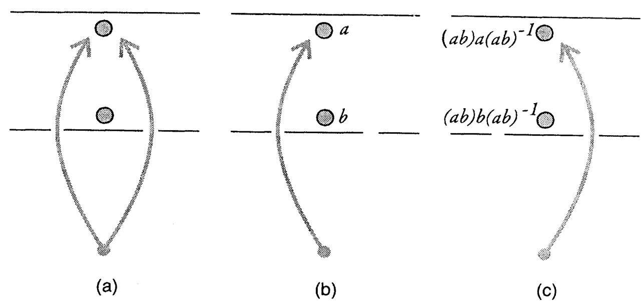

FIG. 3. The charge of a particle can be measured via the Aharonov-Bohm effect in a double-slit interference experiment. (a) When a vortex of known flux b is placed between the two slits, the change in the interference pattern measures $\langle u|D^{(R)}(b)|u\rangle$, where $|u\rangle$ denotes the internal state of the charged particle, and (R) is the representation according to which the charged particle transforms. However, if the charged particle is itself a vortex with flux a, there is a restriction on the charges that can be measured. If the a vortex passes through the left slit, as in (b), it arrives at the screen with flux a, and the vortex between the slits remains in the flux state b. If it passes through the right slit, as in (c), it arrives at the screen with flux $(ab)a(ab)^{-1}$, and the flux of the vortex between the slits becomes $(ab)b(ab)^{-1}$. Thus, no interference is seen if a and b do not commute. Because interference occurs only when a and b commute, this experiment can measure only the transformation properties of the charged projectile under the subgroup $N(a)$ that commutes with a.

Thus, the charged vortices are anyons, and $e^{i\theta_{R^{(a)}}}$ is the anyon phase. A spin-statistics connection holds for these anyons [2, 22], in the sense that an adiabatic interchange of a pair is equivalent to rotating one by 2π—we have $e^{2\pi iJ} = e^{i\theta_{R^{(a)}}}$.

The non-Abelian character of the vortices becomes manifest when we consider combining together two flux-charge composites, and decomposing into states of definite charge. The decomposition has the form

$$|a,R^{(a)}\rangle \otimes |b,R^{(b)}\rangle = \oplus_R |ab,R^{(ab)}\rangle \; , \tag{10}$$

where $R^{(a)}$ denotes an irreducible representation of $N(a)$. The nontrivial problem of decomposing a direct product of a representation of $N(a)$ and a representation of $N(b)$ into a direct sum of representations of $N(ab)$ is elegantly solved by the representation theory of quasitriangular Hopf algebras, as described in the beautiful paper of Bais *et al.* [16] (see also [24, 25]). This decomposition also diagonalizes the monodromy matrix $\mathcal{M} \equiv \mathcal{R}^2$ that acts on the two-vortex state when one vortex winds (counterclockwise) around the other [26, 24]:

$$\mathcal{M} \equiv \mathcal{R}^2 = \exp\left[i\left(\theta_{R^{(ab)}} - \theta_{R^{(a)}} - \theta_{R^{(b)}}\right)\right] \; . \tag{11}$$

Equation (11) follows from Eq. (9) and the spin-statistics connection for anyons, for the action of the monodromy operator, is equivalent to a rotation of the vortex pair by 2π, accompanied by a rotation of each member of the pair by 2π in the opposite sense.

A remarkable property of this decomposition is that a pair of uncharged vortices can be combined together to form an object that carries charge [11, 12, 16]. This is called "Cheshire charge," in homage to the Cheshire cat; the charge can be detected via the Aharonov-Bohm interaction of the pair with another, distant, vortex, but it cannot be localized anywhere on the vortex cores or in their vicinity. Charge can be transferred to or from a pair of vortices due to the Aharonov-Bohm interactions of the pair with another charged object that passes through the two vortices [27, 28, 20]. Since the pair generically carries a fractional spin given by $e^{2\pi iJ} = e^{i\theta_{R(ab)}}$, angular momentum is also transferred in these processes [29].

IV. NON-ABELIAN QUANTUM STATISTICS

In this section, we will briefly describe how the non-Abelian statistics obeyed by non-Abelian vortices fits into general discussions of quantum statistics that have appeared in the literature.

In general discussions of the quantum statistics of indistinguishable particles, the following framework is usually adopted: Suppose that the position of each particle takes values in a manifold M (such as R^d). For n *distinguishable* particles, we would take the classical configuration space to be $M^n = M \times M \times \cdots \times M$. For *indistinguishable* particles (other than bosons), we must restrict the positions so that no two particles coincide, and we must identify configurations that differ by a permutation of the particles. Thus, the classical configuration space becomes

$$\mathcal{C}_n = [M^n - D_n]/S_n \ , \tag{12}$$

where D_n is the subset of M^n in which two or more points coincide, and S_n is the group of permutations of n objects. In general, this configuration space is not simply connected, $\pi_1(\mathcal{C}_n) \neq 0$.

We may now imagine quantizing the theory by using, say, the path integral method. The histories that contribute to the amplitude for a specified initial configuration to propagate to a specified final configuration divide up into disjoint sectors labeled by the elements of $\pi_1(\mathcal{C}_n)$. We have the freedom to weight the contributions from the different sectors with different factors, as long as the amplitudes respect the principle of conservation of probability. In general, there are distinct choices for these weight factors, which correspond to physically inequivalent ways of quantizing the classical theory [30].

We can now define an "exchange operator" that smoothly carries the final particle configuration around a closed path in $\mathcal{C}_n$. Although this exchange does not disturb the positions of the particles, it mixes up the different sectors that contribute to the path integral. Since these sectors are weighted differently, in general, the exchange need not preserve the amplitude. This means that the amplitude need not be a single-valued function of the n positions of the final particles. The effect of the exchange can be expressed as the action of a linear operator acting on the amplitude, and because the total probability sums to 1, this operator is unitary. By considering the effect of two exchanges performed in succession, we readily see that the exchange operators provide a unitary representation of the the group $\pi_1(\mathcal{C}_n)$. Thus, a unitary representation of $\pi_1(\mathcal{C}_n)$ acting on amplitudes (or wave functions) is a general feature of the quantum mechanics of n indistinguishable particles. [The weight factors appearing in the path integral also transform as a unitary representation of $\pi_1(\mathcal{C}_n)$.]

If the manifold is R^d for $d \geq 3$, then $\pi_1(\mathcal{C}_n) = S_n$, and the exchange operators provide a unitary representation of the permutation group S_n. In addition to the familiar one-dimensional representations associated with Bose and Fermi statistics, non-Abelian representations ("parastatistics") are also possible in principle. But it is known that, in a local quantum field theory, parastatistics can always be reduced to Bose or Fermi statistics by introducing additional degrees of freedom and a suitable global symmetry that acts on these degrees of freedom [31]. For $d = 1$, in this framework, no exchange is possible—the particles cannot pass through each other—and there is no quantum statistics to discuss.

The case $d = 2$ is the most interesting. Then $\pi_1(\mathcal{C}_n)$ is B_n, the braid group on n strands. This is an infinite group with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, where σ_j may be interpreted as a (counterclockwise) exchange of the particles in positions j and $j+1$. These generators obey the defining relations

$$\sigma_j \sigma_k = \sigma_k \sigma_j \ , \quad |j-k| \geq 2 \ , \tag{13}$$

and

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \ , \quad j = 1, 2, \ldots, n-2 \tag{14}$$

(the Yang-Baxter relation). It follows from the Yang-Baxter relation that, in a one-dimensional unitary representation of the braid group, all of the σ_j's are represented by a common phase $e^{i\theta}$. This is anyon statistics. But non-Abelian representations of the braid group may also arise in local quantum field theories. Indistinguishable particles in two dimensions that transform under exchange as a non-Abelian unitary representation of the braid group are said to obey non-Abelian statistics.

Our discussion of non-Abelian vortices fits into the general framework outlined above, but with an important caveat. If the vortex flux takes values in an unbroken local symmetry group $H(x_0)$, we treat two vortices with flux a and b as "indistinguishable" if $b = hah^{-1}$ for some $h \in H(x_0)$ and if both vortices have the same charge [transform as the same irreducible representation of the centralizer $N(a) \cong N(b)$]. The philosophy is that the particles are regarded as indistinguishable if an exchange of the particles can conceivably occur (in the presence of other particles with suitable quantum numbers) without changing the quantum numbers assigned to the many-particle configuration. The caveat is that these "indistinguishable" particles are not really identical. For example, two vortices with flux a and b are distinct—e.g., the a vortex will not annihilate the antiparticle of the b vortex—if $a \neq b$, even if a and b are in the same conjugacy class.

This classification of the different types of "indistinguishable" vortices can also be described in terms of the representation theory of a quasitriangular Hopf algebra, or "quantum double" [16, 24, 25]. The quantum double $D(H)$ associated with a finite group H is an algebra that is generated by global gauge transformations and projection operators that pick out a particular value of the flux. A basis for the algebra is[2]

$$\{P_h a\ , \quad h\,, a \in H\}\ , \tag{15}$$

where P_h projects out the flux value h, and a is a gauge transformation. Since the projection operators satisfy the relations

$$P_h P_g = \delta_{h,g} P_h\ , \quad a P_h a^{-1} = P_{aha^{-1}}\ , \tag{16}$$

the multiplication law for the algebra can be expressed as

$$(P_h a)\,(P_g b) = \delta_{h,aga^{-1}}\,(P_h ab)\ . \tag{17}$$

An irreducible representation of the quantum double $D(H)$ can be labeled $([a], R^{(a)})$, where $[a]$ denotes the conjugacy class that contains $a \in H$ and $R^{(a)}$ is an irreducible representation of the centralizer $N(a)$ of a. This is the induced representation of $D(H)$ generated by the representation $R^{(a)}$ of $N(a)$. The space on which this representation acts is a space of charged vortex states that transform irreducibly under the global gauge transformations. In order for an exchange contribution to an amplitude to interfere with the direct amplitude, the two vortices being exchanged must belong to the same irreducible representation of the quantum double.

[If a Chern-Simons term is added to the action of the underlying gauge theory, the situation becomes somewhat more complicated [16]. The Chern-Simons term distorts the charge spectrum of vortices with a specified value of the flux, and unremovable phases can enter the multiplication law of the quantum double [16,24,32]. The vortex states may then transform as a projective (ray) representation under gauge transformations.]

Consider a state of n "indistinguishable" vortices, all with flux conjugate to a, and all transforming as the representation $R^{(a)}$ of the centralizer $N(a)$ [in other words, all of the vortices belong to the irreducible representation $([a], R^{(a)})$ of the quantum double]. A basis for these states can be constructed, in which, at each vortex position, we assign a definite flux, and a definite basis state in the vector space on which the representation $R^{(a)}$ acts. Under exchange, these states transform as a representation of B_n that is in general non-Abelian and reducible. This reducible representation can be decomposed into irreducible components. Each irreducible component describes an n-particle state obeying definite "braid statistics."

The point that we wish to emphasize is that the exchange operator will typically modify the quantum numbers that are assigned to the n particle positions. Thus, physical observables, such as transition probabilities or cross sections, need not be invariant under exchange. Instead, the exchange relates the value of the observable for one assignment of quantum numbers to the particle positions to the value of the observable for another choice of quantum numbers. Correspondingly, as we stressed above, the observables are not single valued functions of the particle positions. Only a subgroup of the braid group returns the quantum numbers to their original values, and so preserves the values of the physical observables. (It is possible to restore the single-valuedness of the many-body wave functions by introducing on the configuration space a suitable connection with nontrivial holonomy. The existence of such a connection does not alter the essential physical point, which is that "indistinguishable" vortices may have distinct quantum numbers that can really be measured by an observer.)

Even distinguishable vortices have non-trivial Aharonov-Bohm interactions, and so it is appropriate to broaden this framework slightly. We may consider a many-particle state containing n_1 particles of type 1 [with the type characterized by the class of the flux a, and the charge $R^{(a)}$, or, in other words, by the irreducible representation $([a], R^{(a)})$ of the quantum double], n_2 particles of type 2, and so on. Then an exchange of two particles is permitted only if the particles are of the same type, and the wave function transforms as a unitary representation of the "partially colored braid group" $B_{n_1,n_2,\ldots}$ [29, 33].

Within this framework, a general connection between spin and statistics can be derived, assuming the existence of an antiparticle corresponding to each particle [34, 22, 23]. The essence of the connection is that, if two particles are truly *identical* (carry *exactly* the same quantum numbers), then an exchange of the two particles can be smoothly deformed to a process in which no exchange occurs, but one of the particles rotates by 2π [34]. (The reason that the quantum numbers must be the same is that, for the deformation to be possible, it is necessary for the antiparticle of the first particle to be able to annihilate the second particle.) It follows from the connection between spin and statistics that the effect of an exchange of two objects that are truly identical must be to modify the many-body wave function by the phase $e^{2\pi i J}$, where J is the spin of the object. We have already remarked in Sec. III that this is true for non-Abelian vortices with the same flux and charge. Thus, we find that non-Abelian statistics is perfectly compatible with the connection between spin and statistics.

There are deep connections between the theory of indistinguishable particles in two spatial dimensions and conformally invariant quantum field theory in two-dimensional *spacetime*. These connections have been explored most explicitly in the case of $(2+1)$-dimensional topological Chern-Simons theories [35], but appear to be more general [23]. There is a close mathematical analogy between the *particle* statistics in two spatial dimensions that we have outlined here and the *field* statistics in two-dimensional conformal field theory. In the latter case, all correlation functions can be constructed by assembling "conformal blocks," and the conformal blocks typically transform as a nontrivial unitary representation of the braid group when the arguments of the correlation func-

[2]In Refs. [16, 24], the notation $^{h}\underline{|\;\;}_{a}$ is used for $P_h a$.

tion are exchanged. (See Ref. [36] for a review.) However, in discussions of conformal field theory, it is usually the case that observables of interest (the correlation functions themselves) are invariant under exchange.

V. VORTEX-VORTEX SCATTERING

The holonomy interaction between vortices induces Aharonov-Bohm vortex-vortex scattering, as pointed out by Wilczek and Wu [6] and Bucher [7]. Suppose that a vortex that initially carries flux b is incident on a fixed vortex that initially carries flux a. Let us suppose, for now, that the vortices are uncharged.

To understand the behavior of the b vortex propagating on the background of the fixed a vortex, it is convenient to adopt a path integral viewpoint. Consider the two possible paths shown in Fig. 4. If the vortex follows the path that passes below the scattering center, it will arrive at its destination with flux b. But if it follows the path that passes above the scattering center, it arrives carrying the flux aba^{-1}. Thus, if the flux of the scattering center and the flux of the projectile do not commute, the contribution to the path integral from paths that pass below does not interfere with the contribution from paths that pass above. Therefore, a plane wave propagating on the background of the fixed vortex does not remain a plane wave—there is nontrivial scattering.

More generally, the paths can be classified according to how many times they wind around the scattering center (relative to some standard path). The flux of a b vortex that winds around an a vortex k times is modified according to

$$|b\rangle \rightarrow |(ab)^k b(ab)^{-k}\rangle \equiv |k\rangle \ . \tag{18}$$

Since the unbroken gauge group H is assumed to be finite, the flux eventually returns to its original value, say, after n windings.

The flux of the scattered vortex, then, can take any one of n values. The amplitude for the vortex to arrive at the detector in the flux state $|k\rangle$ defined in Eq. (18) can be found by summing over all paths with winding number congruent to k modulo n. Since only every nth winding sector is included in the amplitude ψ_k for flux channel k, this amplitude is not a periodic function of the polar angle ϕ with period 2π; rather, the period is $2\pi n$. The n amplitudes are related by the nontrivial monodromy property

$$\psi_k(r, \phi + 2\pi) = \psi_{k+1}(r, \phi) \tag{19}$$

[where $\psi_{k+n}(r, \phi) \equiv \psi_k(r, \phi)$]. Similarly, the exclusive cross section for flux channel k is also multivalued:

$$\sigma_k(\theta - 2\pi) = \sigma_{k+1}(\theta) \ , \tag{20}$$

where $\theta = \pi - \phi$ is the scattering angle. The inclusive cross section

$$\sigma_{\rm inc}(\theta) = \sum_{k=0}^{n-1} \sigma_k(\theta) \tag{21}$$

is single valued.

As we stressed in the Introduction, the multivaluedness of the exclusive cross sections is natural and unavoidable in this context. Whenever we assign a flux to a non-Abelian vortex, we are implicitly adopting a conventional procedure for measuring the flux. For example, the procedure might be to carry the vortex to the "vortex bureau of standards" and analyze it there by performing Aharonov-Bohm interference experiments with various charged particles. Then the multivaluedness arises because, if we carry a vortex in the flux state $|k\rangle$ once around the scattering center (counterclockwise) before returning it to the bureau of standards, the analysis will identify it as the flux state $|k+1\rangle$.

For each value of the scattering angle, we might choose a standard path along which the vortex is to be returned to the bureau for analysis after the scattering event. For example, we might decide to carry it home through the upper half plane for $\theta \in [0, \pi)$ and through the lower half plane for $\theta \in [-\pi, 0)$, as shown in Fig. 5. Then the exclusive cross sections are single valued, but are discontinuous at $\theta = 0$:

$$\sigma_k(\theta = 0^+) = \sigma_{k+1}(\theta = 0^-) \ . \tag{22}$$

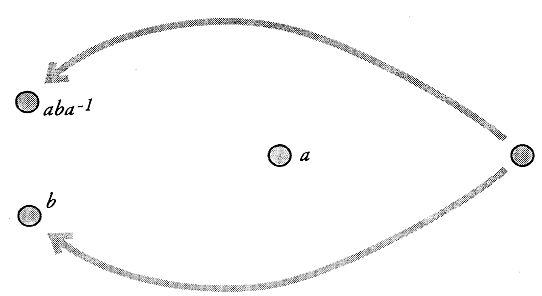

FIG. 4. Two paths that contribute to the amplitude for a b vortex propagating on the background of a fixed a vortex. If the b vortex passes below the a vortex, it arrives at its destination with flux b; if it passes above the a vortex, it arrives at its destination with flux aba^{-1}. Thus, these two paths do not interfere if a and b do not commute.

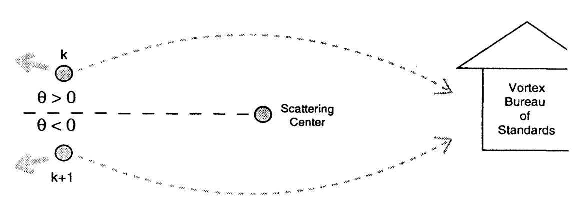

FIG. 5. A convention for measuring the flux of a scattered vortex that is single valued but discontinuous. If the vortex is scattered into the upper half plane ($0 < \theta < \pi$), it is carried back to the "vortex bureau of standards" above the scattering center; if the vortex is scattered into the lower half plane ($-\pi < \theta < 0$), it is carried back to the bureau below the scattering center. With this convention, the scattering cross section is discontinuous at $\theta = 0$; the cross section in the "k" channel at $\theta = 0^+$ matches the cross section in the "$k+1$" channel at $\theta = 0^-$.

The choice of a standard path amounts to an arbitrary restriction of the n-valued exclusive cross sections to a single branch.

In a sense, the multivaluedness of the wave functions, and of the exclusive cross sections, arises because we have insisted on expressing the flux of the vortices in terms of a multivalued basis—that basis defined by parallel transport of the flux in the background gauge potential of the scattering center. The propagation of the projectile on this background is really nonsingular, and the multivaluedness of the amplitudes actually compensates for the multivaluedness of the basis. This is quite analogous to the "singular-gauge" description of ordinary Abelian Aharonov-Bohm scattering. There, expressing the phase of the electron wave function relative to a basis defined by parallel transport is equivalent to performing a singular-gauge transformation that gauges away the vector potential and introduces a discontinuity in the wave function. The difference in the non-Abelian case is that the discontinuity corresponds to a jump in observable quantum numbers of the projectile, as explained above. It is natural to use the multivalued basis because it reflects what a team of experimenters would really find if they brought their detectors together to calibrate them alike.

Mathematically, finding the Aharonov-Bohm amplitude for a vortex propagating on the background of a fixed vortex is equivalent to finding the amplitude for a free particle propagating on an n-sheeted surface. [The closely related problem of a free particle propagating on a cone has been discussed in connection with (2 + 1)-dimensional general relativity [17, 18].] The most convenient way to solve the problem is to transform to a basis of "monodromy eigenstates," since for the elements of this basis the scattering reduces to Abelian Aharonov-Bohm scattering. If the ψ_k's obey the monodromy property Eq. (19), then the monodromy eigenstate basis is

$$\chi_l = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i k l/n} \psi_k \ , \tag{23}$$

with the property

$$\chi_l(r, \phi + 2\pi) = e^{2\pi i l/n} \chi_l(r, \phi) \ . \tag{24}$$

These monodromy eigenstates correspond to states of the two-vortex system that have definite charge, in the sense that they are eigenstates of the gauge transformation $ab \in H$, where ab is the total flux.

We may think of the wave functions χ_l as the coefficients in an expansion of a single-valued wave function in a multivalued basis. That is, we can express a single-valued wave function as

$$|\psi\rangle = \sum_{r,\phi,l} |r, \phi, l\rangle\langle r, \phi, l|\psi\rangle \ , \tag{25}$$

where the basis $|r, \phi, l\rangle$ is "twisted" according to

$$|r, \phi + 2\pi, l\rangle = e^{-2\pi i l/n} |r, \phi, l\rangle \ . \tag{26}$$

The coefficients $\chi_l(r, \phi) = \langle r, \phi, l|\psi\rangle$ inherit the property Eq. (24) from the property Eq. (26) of the basis.

By standard methods [37], we can find the solution to the free-particle nonrelativistic Schrödinger equation that obeys the condition

$$\chi_\alpha(\phi + 2\pi) = e^{2\pi i \alpha} \chi_\alpha(\phi) \ , \quad 0 \le \alpha < 1 \ , \tag{27}$$

and matches a plane wave incoming from $\phi = 0$. The asymptotic large-r behavior of this solution is

$$\chi_\alpha \sim e^{-i\mathbf{p}\cdot\mathbf{x}} + \frac{e^{ipr}}{\sqrt{r}} f_\alpha(\phi) \ , \quad -\pi < \phi < \pi \ , \tag{28}$$

where

$$f_\alpha(\phi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{1}{1 + e^{i\phi}} \right) e^{i\alpha\phi} \left(e^{-i\alpha\pi} - e^{i\alpha\pi} \right) \ , \qquad 0 \le \alpha < 1 \ . \tag{29}$$

Here $e^{-i\alpha\pi}$ is the phase shift for the partial waves with non-negative integer part of the orbital angular momentum and $e^{i\alpha\pi}$ is the phase shift for the partial waves with negative integer part of the orbital angular momentum. The semiclassical interpretation is that wave packets that pass above and below the scattering center acquire a relative phase $e^{2\pi i\alpha}$, the Aharonov-Bohm phase.

There are two subtleties concerning Eqs. (28) and (29) that deserve comment. The first subtlety (which is not very important for what follows) is that there is an order of limits ambiguity in the evaluation of the amplitude—the limit $r \to \infty$ does not commute with the limit $\phi \to \pm\pi$ [38]. In Eqs. (28) and (29), we have taken $r \to \infty$ for fixed ϕ between $-\pi$ and π. Thus, χ_α actually satisfies Eq. (27), although the first term in the asymptotic form Eq. (28) appears not to. [For large r, the phase of the plane wave in Eq. (27) suddenly advances by $e^{2\pi i\alpha}$ as ϕ increases through a narrow wedge near $\phi = \pi$. Of course, if we construct localized wave packets, then the unscattered wave has support at $\phi = 0, \pm\pi$ as $r \to \infty$, and the form of the plane wave away from the forward direction is of no consequence anyway.] The second subtlety, which is very important for what follows, concerns the α dependence of the amplitude. The monodromy condition Eq. (27) depends only on $\alpha - [\alpha]$, where $[\alpha]$ denotes the greatest integer less than or equal to α. Thus, as one can explicitly verify, the amplitude $f_\alpha(\phi)$, when α is not restricted to lie in the range $[0, 1)$, takes the same form as Eq. (29), but with α replaced by $\alpha - [\alpha]$. The somewhat surprising feature is that, as a function of α, $f_\alpha(\phi)$ is not differentiable when α is an integer.

The form Eq. (29) for the scattering amplitude in the monodromy eigenstate basis is readily generalized to an arbitrary basis, if we express it in terms of the braid operator $\mathcal{R}$, the square root of the monodromy operator $\mathcal{M}$. The general monodromy condition satisfied by the wave function can be expressed as

$$\psi(\phi + 2\pi) = \mathcal{M}\psi(\phi) \ , \tag{30}$$

where $\mathcal{M}$ is a unitary matrix acting on internal indices. Then the basis-independent form for the scattering amplitude is

$\langle \text{out}|f(\phi)|\text{in}\rangle$

$$= \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{1}{1+e^{i\phi}} \right) \langle \text{out}| \mathcal{R}^{\phi/\pi} \left(\mathcal{R}^{-1} - \mathcal{R} \right) |\text{in}\rangle \,, \quad (31)$$

where $\mathcal{R}$ is defined by $\mathcal{R}^2 = \mathcal{M}$, and $|\text{in}\rangle$ and $|\text{out}\rangle$ denote the incoming and outgoing wave functions in internal space, respectively. This definition of $\mathcal{R}$ leaves an ambiguity in $\mathcal{R}^{(\phi/\pi \mp 1)}$, and it is important to resolve this ambiguity correctly. Acting on an eigenstate of $\mathcal{M}$ with

$$\mathcal{M} = e^{2\pi i\alpha} \,, \quad (32)$$

we define

$$\mathcal{R}^{(\phi/\pi\mp 1)} \equiv e^{i(\alpha - [\alpha])(\phi \mp \pi)} \,. \quad (33)$$

In Eq. (31), the state $|\text{in}\rangle$ is expressed in terms of an arbitrary basis, and we have assumed that the state $|\text{out}\rangle$ is expressed in terms of a basis that is obtained by parallel transport of the in basis. This out basis is multivalued, and so we have in effect evaluated the amplitude in a "singular gauge."

From Eq. (31), we obtain the cross section

$$\sigma_{\text{in}\to\text{out}}(\phi) = |f(\phi)|^2 = \frac{1}{2\pi p} \left(\frac{1}{4\cos^2\phi/2} \right) \left| \langle \text{out}| \mathcal{R}^{\phi/\pi} \left(\mathcal{R}^{-1} - \mathcal{R} \right) |\text{in}\rangle \right|^2 \,. \quad (34)$$

By summing $|\text{out}\rangle$ over a complete basis, we obtain the inclusive cross section

$$\sigma_{\text{in}\to\text{all}}(\theta) = \frac{1}{2\pi p} \left(\frac{1}{\sin^2\theta/2} \right) \frac{1}{2} \left(1 - \text{Re}\langle \text{in}|\mathcal{R}^2|\text{in}\rangle \right) \,, \quad (35)$$

where $\theta = \pi - \phi$ is the scattering angle; this is the formula derived by Verlinde [15].

For monodromy eigenstates with $\mathcal{M} = e^{2\pi i\alpha}$, Eq. (34) reduces to the familiar form of the Aharonov-Bohm cross section:

$$\sigma_\alpha(\theta) = \frac{1}{2\pi p} \left(\frac{\sin^2 \pi\alpha}{\sin^2\theta/2} \right) \,, \quad (36)$$

which is a single-valued function of the scattering angle. But the recurring theme of this paper is that it is often convenient to express the scattering states in terms of a basis other than the monodromy eigenstate basis. Then the exclusive cross sections are in general multivalued, but the inclusive cross section (summed over all possible final state quantum numbers) is always singlevalued.

Returning to the special case of (uncharged) vortex-vortex scattering, we obtain the amplitude in the flux eigenstate basis by coherently summing the monodromy eigenstate amplitudes with appropriate phases:

$$\begin{aligned} \langle k|f(\phi)|k=0\rangle &= \frac{1}{n}\sum_{l=0}^{n-1} e^{2\pi ikl/n} f_{l/n}(\phi) \\ &= \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{i}{2n} \right) \frac{\sin(\pi/n)}{\sin\left\{\frac{1}{2n}\left[\phi + (2k+1)\pi\right]\right\} \sin\left\{\frac{1}{2n}\left[\phi + (2k-1)\pi\right]\right\}} \,. \end{aligned} \quad (37)$$

This formula has the expected monodromy property

$$\langle k|f(\phi + 2\pi)|k=0\rangle = \langle k+1|f(\phi)|k=0\rangle \,. \quad (38)$$

[Equation (37) is actually a special case of the formula derived in (2+1)-dimensional gravity by 't Hooft [17] and Deser and Jackiw [18].]

This amplitude has the infinite forward peak that is characteristic of Aharonov-Bohm scattering. For $\phi = \pi$, the infinite peak occurs in the flux channels $k = 0, -1$, and for $\phi = -\pi$, it occurs in the channels $k = 1, 0$. For ϕ near π, the leading behavior of the amplitude is

$$\begin{aligned} \langle k=0|f(\phi)|k=0\rangle &\sim -\langle k=-1|f(\phi)|k=0\rangle \\ &\sim \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{i}{\phi - \pi} \right) \,. \end{aligned} \quad (39)$$

This leading behavior has a simple interpretation. From a path integral viewpoint, the forward peak is generated by paths that pass above or below the scattering center with a large impact parameter, without any winding around the center. If the projectile passes above, it is detected near $\phi = \pi$ as a $k = 0$ vortex (or near $\phi = -\pi$ as a $k = 1$ vortex); if it passes below, it is detected near $\phi = \pi$ as a $k = -1$ vortex (or near $\phi = -\pi$ as a $k = 0$ vortex). Near $\phi = \pi$, the amplitude in the $k = 0, -1$ channels is equivalent to the diffraction pattern generated by a "sharp edge," since paths that wind n times around the scattering center make a negligible contribution. The near-forward amplitude in the $k = 0$ channel comes from summing all of the partial waves with non-negative angular momentum, and the near-forward amplitude in the $k = -1$ channel comes from summing the partial waves with negative angular momentum. Thus, the forward peak in each channel is half as strong as the forward peak for "maximal" ($\alpha = 1/2$) Abelian Aharonov-Bohm scattering.

The inclusive cross section (obtained by summing over all possible final flux channels) can be immediately read off from Eq. (35). If the projectile is a flux eigenstate, and the scattering center is a flux eigenstate whose flux does not commute with that of the projectile, then we have $\langle \text{in}|\mathcal{R}^2|\text{in}\rangle = 0$, and the inclusive cross section takes the universal form

$$\sigma_{\text{flux eigenstate}\to\text{all}}(\theta) = \left(\frac{1}{2}\right)\frac{1}{2\pi p}\left(\frac{1}{\sin^2\theta/2}\right) , \qquad (40)$$

that is, half the cross section for maximal Aharonov-Bohm scattering.

So far, we have assumed that the vortex that is being scattered carries no charge. Let us briefly comment on how the analysis is modified when the scattered vortex is charged.

Suppose that the vortex with flux a transforms as some irreducible representation $D^{R^{(a)}}$ of $N(a)$, and that the vortex with flux b transforms as some irreducible representation $D^{R^{(b)}}$ of $N(b)$. And suppose as before that the fluxes return to their original values after the monodromy operator acts n times (that is, after the b vortex winds around the a vortex n times). For charged vortex states, although $\mathcal{M}^n$ preserves the flux values, it acts on the vortex pair as a nontrivial $N(a)\otimes N(b)$ transformation. Specifically, we have

$$\mathcal{M}^n : |a\rangle \otimes |b\rangle$$
$$\to D^{R(a)}[(ab)^n a^{-n}]|a\rangle \otimes D^{R(b)}[(ab)^n b^{-n}]|b\rangle . \qquad (41)$$

Note that, since by assumption $(ab)^n a (ab)^{-n} = a$ and $(ab)^n b (ab)^{-n} = b$ (because $\mathcal{M}^n$ preserves the fluxes), $(ab)^n a^{-n} \in N(a)$ and $(ab)^n b^{-n} \in N(b)$.

For the case of scattering a b vortex off of a fixed a vortex, we consider the states $|k\rangle$ defined by

$$|k\rangle \equiv \mathcal{M}^k|b\rangle , \quad k = 0,1,2,\ldots,n-1 , \qquad (42)$$

with

$$\mathcal{M}^n|k=0\rangle = D^{R^{(b)}}[(ab)^n b^{-n}]|k=0\rangle . \qquad (43)$$

To diagonalize the monodromy operator, we first diagonalize the unitary transformation $D^{R^{(b)}}[(ab)^n b^{-n}]$. Corresponding to each eigenstate of this operator with eigenvalue $e^{2\pi i\beta}$ is a set of monodromy eigenstate wave functions

$$\chi_{l,\beta} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi i k(l+\beta)/n}\psi_{k,\beta} , \qquad (44)$$

with the property

$$\chi_{l,\beta}(r,\phi+2\pi) = e^{2\pi i(l+\beta)/n}\chi_{l,\beta}(r,\phi) . \qquad (45)$$

For particular charged states with specified flux, we may evaluate Eq. (31) by coherently superposing the Aharonov-Bohm amplitudes for these monodromy eigenstates.

VI. INDISTINGUISHABLE VORTICES

The effects of quantum statistics can be seen in the two-body scattering of indistinguishable particles, because exchange scattering can occur; it is possible to lose track of "who's who." In the case of non-Abelian vortices, the exchange effects are more subtle than for Abelian anyons—in general, whether two vortices behave like identical or distinct particles when they are brought together depends on their *history*. Suppose that two identical vortices each carry the flux $a \in H$. If one of the vortices should voyage around another vortex with flux b, and then return to its partner, it would then carry flux bab^{-1}. Hence, if a and b do not commute, it would now be distinct from the other a vortex.

For exchange effects to occur in vortex-vortex scattering, the braid operator must have an orbit of odd order acting on the two vortex state. That is, $\mathcal{R}^n$ must preserve the two-vortex state for some odd n. If so, there will be a contribution to the vortex-vortex scattering amplitude in which the two vortices change places that interferes with the direct amplitude.

As a simple example, consider the permutation group on three objects S_3, where the fluxes are two distinct two-cycles. Then the braid operator defined by Eq. (7) has the orbit

$$\begin{aligned}\mathcal{R}: \quad &|(12),(23)\rangle \to |(13),(12)\rangle \\ &\to |(23),(13)\rangle \\ &\to |(12),(23)\rangle , \end{aligned} \qquad (46)$$

of order 3. (See Fig. 6.) Thus, there is an exchange contribution to the scattering of a (12) vortex and a (23) vortex. (In this case, the centralizer of the total flux is Z_3, and the braid eigenstates are the linear combinations of these three states that have definite Z_3 charge.)

Two vortices whose flux belongs to the same conjugacy class of the unbroken group H have the same mass, and we can easily derive a formula for the vortex-vortex scattering amplitude in the center-of-mass frame, using the same methods as in the previous section. This formula will incorporate the exchange effects whenever the braid operator has an odd orbit acting on the two-vortex state. The two-body wave function in the center-of-mass frame

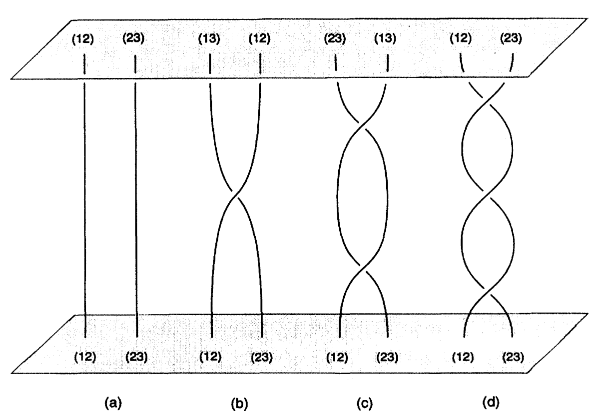

FIG. 6. Paths contributing to the amplitude for the propagation of a pair of vortices. The initial vortices carry flux taking the values (12) and (23) in S_3. If the vortices braid once as in (b) or twice as in (c), the quantum numbers of the pair are modified. But if the vortices braid three times as in (d), the final quantum numbers match the initial quantum numbers. Thus, paths (a) and (d) add coherently in the amplitude, although the two vortices change places.

will now have the property

$$\psi(r,\phi+\pi) = \mathcal{R}\psi(r,\phi) \; , \tag{47}$$

where the braid operator $\mathcal{R}$ is a unitary matrix acting on the internal indices of the wave function. The problem is to solve the free-particle Schrödinger equation subject to this condition.

If the two-body state is a "braid eigenstate,"

$$\chi_\alpha(r,\phi+\pi) = e^{i\pi\alpha}\chi_\alpha(r,\phi) \; , \quad 0 \le \alpha < 2 \; , \tag{48}$$

then the problem is equivalent to anyon-anyon scattering, with statistical phase $e^{i\theta} = e^{i\pi\alpha}$. We can find the solution to the free-particle Schrödinger equation that obeys Eq. (48) and matches plane waves coming from $\phi = 0$ and $\phi = \pi$. The asymptotic large-r behavior of this solution is [39]

$$\chi_\alpha \sim \left(e^{-i\mathbf{p}\cdot\mathbf{x}} + e^{i\alpha\pi}e^{i\mathbf{p}\cdot\mathbf{x}}\right) + \frac{e^{ipr}}{\sqrt{r}} f_\alpha(\phi) \; , \quad 0 < \phi < \pi \; , \tag{49}$$

where

$$f_\alpha(\phi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{2}{1-e^{2i\phi}}\right) e^{i\alpha\phi}\left(e^{-i\alpha\pi} - e^{i\alpha\pi}\right) \; , \quad 0 \le \alpha < 2 \; . \tag{50}$$

[As in our discussion of scattering off a fixed target, we remark that the limit $r \to \infty$ does not commute with the limit $\phi \to 0, \pi$ [38]. Thus, χ_α actually satisfies Eq. (48), although the first term in the asymptotic form Eq. (49) appears not to.] In an arbitrary basis, in which the braid operator is not necessarily diagonal, we have

$$\langle \text{out}|f(\phi)|\text{in}\rangle = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{2}{1-e^{2i\phi}}\right) \langle \text{out}|\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right)|\text{in}\rangle \; , \tag{51}$$

where $|\text{in}\rangle$ and $|\text{out}\rangle$ denote the incoming and outgoing two-body wave functions in internal space, respectively. As in our discussion of scattering off of a fixed center, there is an ambiguity in the evaluation of $\mathcal{R}^{(\phi/\pi\mp 1)}$, and we must now resolve this ambiguity slightly differently than before. If α is not restricted to the range $[0, 2)$, then α must be replaced by $\alpha - [[\alpha]]$ in Eq. (50), where $[[\alpha]]$ denotes the greatest *even* integer less than or equal than α. Thus, acting on an eigenstate of $\mathcal{R}$ with eigenvalue

$$\mathcal{R} = e^{i\pi\alpha} \; , \tag{52}$$

we define $\mathcal{R}^{(\phi/\pi\mp 1)}$ by

$$\mathcal{R}^{(\phi/\pi\mp 1)} = e^{i(\alpha-[[\alpha]])(\phi\mp\pi)} \; . \tag{53}$$

The cross section is

$$\sigma_{\text{in}\to\text{out}}(\phi) = |f(\phi)|^2 = \frac{1}{2\pi p}\left(\frac{1}{\sin^2\phi}\right)\left|\langle \text{out}|\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1}-\mathcal{R}\right)|\text{in}\rangle\right|^2 \; . \tag{54}$$

By summing $|\text{out}\rangle$ over a complete basis, we obtain the inclusive cross section

$$\sigma_{\text{in}\to\text{all}}(\theta) = \frac{1}{2\pi p}\left(\frac{1}{\sin^2\theta}\right) 2\left(1 - \text{Re}\langle \text{in}|\mathcal{R}^2|\text{in}\rangle\right) \; , \tag{55}$$

where $\theta = \pi - \phi$ is the scattering angle.

The general problem can be solved by expressing the two-body state as a linear combination of braid eigenstates, and then coherently superposing the anyon-anyon amplitudes. In the case of (uncharged) vortex-vortex scattering, if the initial state is a vortex with flux a coming from $\phi = \pi$ and a vortex with flux b coming from $\phi = 0$, then let us denote by $|k\rangle$ the state obtained when the braid operator $\mathcal{R}$ defined by Eq. (7) acts on the initial state k times:

$$|k\rangle \equiv \mathcal{R}^k|a,b\rangle \; . \tag{56}$$

Suppose that the two-vortex state returns to the initial state after $\mathcal{R}$ acts n times. (Note that, in a departure from the notation of the previous section, k and n now denote the number of times the *braid* operator acts on the initial state, rather than the monodromy operator $\mathcal{M} = \mathcal{R}^2$.) Then,

$$\chi_{2l/n} = \sum_{k=0}^{n-1} e^{-2\pi ikl/n}|k\rangle \tag{57}$$

is a braid eigenstate with eigenvalue $e^{i\pi\alpha} = e^{2\pi il/n}$, and the scattering amplitude in the flux eigenstate basis is

$$\begin{aligned}\langle k|f(\phi)|k=0\rangle &= \frac{1}{n}\sum_{l=0}^{n-1} e^{2\pi ikl/n} f_{2l/n}(\phi) \\ &= \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{i}{n}\right)\frac{\sin(2\pi/n)}{\sin\left\{\frac{1}{n}\left[\phi+(k+1)\pi\right]\right\}\sin\left\{\frac{1}{n}\left[\phi+(k-1)\pi\right]\right\}} \; .\end{aligned} \tag{58}$$

This formula has the desired property

$$\langle k|f(\phi+\pi)|k=0\rangle = \langle k+1|f(\phi)|k=0\rangle \; . \tag{59}$$

Equation (58) applies for any value of n, but there is an exchange contribution to the amplitude only for odd n. [Note that, if n and k are even, Eq. (58) precisely coincides with Eq. (37), as one would expect.]

The amplitude has the expected infinite peak at $\phi = \pi$ in the channels $k = 0, -2$ and at $\phi = 0$ in the channels $k = \pm 1$. As in our discussion of scattering off of a fixed center, these peaks are generated by paths in which the two vortices pass one another with a large impact parameter, without any winding. If the vortex incident from the right passes above the vortex incident from the left, then, with our conventions, a $k = 0$ state is detected near $\phi = \pi$, and a $k = 1$ state is detected near $\phi = 0$. If the vortex incident from the right passes below, then a $k = -2$ state is detected near $\phi = \pi$, and a $k = -1$ state is detected near $\phi = 0$.

VII. CONTINUOUS SYMMETRY: THE ALICE VORTEX

So far, we have assumed that the unbroken local symmetry group is a discrete group. In this section, we will briefly consider the properties of non-Abelian vortices when the gauge group is continuous.

If the unbroken gauge group has a non-Abelian Lie algebra, then the gauge interaction is presumably confining. In fact, even if the Lie algebra is Abelian [a product of U(1)'s], then charge is logarithmically confined in two spatial dimensions. That is, the Coulomb energy of a charged object is logarithmically infrared divergent. Nevertheless, we might be interested in the Aharonov-Bohm interactions of vortices and charged particles on distance scales that are small compared to the confinement scale or under circumstances where the Coulomb energy can be safely neglected.

Strictly speaking, there is no Aharonov-Bohm amplitude for the scattering of a charged particle off of a vortex, because there are no asymptotic charged states. Still, the formalism discussed in this paper finds some application. We can imagine placing a compensating charge far away from the scattering center, and consider the scattering of a wave packet in a bounded region that is small compared to the distance to the compensating charge (or small compared to the confinement distance scale). Furthermore, the charge of a particle behaves like $\hbar e$, where e is a (classical) gauge coupling, and so Coulomb effects are of order $(\hbar e)^2$ and are higher order corrections to Aharonov-Bohm scattering in the semiclassical (small $\hbar$) limit. Under suitable conditions, the deflection of the wave packet is described to good accuracy by our general formula for the Aharonov-Bohm amplitude, Eq. (31).

The case of vortex-vortex scattering is more complicated. We can imagine scattering two vortices that are flux eigenstates. (More properly, in the case of continuous gauge symmetry, we should consider narrow "flux wave packets," superpositions of flux eigenstates with small dispersion.) However, a pair of flux eigenstates does not have definite charge; when the state of the pair is decomposed into charge eigenstates, the states of nonzero charge have infrared divergent Coulomb energy. Again, there is a need for a compensating charge. But in this case, the value of the compensating charge must be correlated with the state of the vortex pair. If we trace over the state of the compensating charge, we obtain a density matrix for the vortex pair that is an *incoherent* superposition of charge eigenstates. Thus, the "scattering cross section" is an incoherent sum of the cross sections for the various charge (or braid) eigenstates, and Eq. (31) does not apply.

To make the discussion more definite, let us consider the simplest model that exhibits these features, the "Alice" model [10–13]. The unbroken symmetry group in this case is the semidirect product of U(1) with Z_2. The group has a component connected to the identity, the U(1) subgroup, that can be parametrized as

$$\{e^{i\omega Q}\ , \quad 0 \le \omega < 2\pi\}\ , \tag{60}$$

where $Q = \sigma_3$ is the U(1) generator. There is also a component that is not connected to the identity:

$$\{i\sigma_2 e^{i\omega Q}\ , \quad 0 \le \omega < 2\pi\}\ . \tag{61}$$

Each element of the disconnected component anticommutes with Q. Thus, the Alice model can be characterized as a generalization of electrodynamics in which charge conjugation is a *local* symmetry.

An "Alice vortex" carries flux that takes a value in the disconnected component of this group. The monodromy operator associated with transport around this vortex, acting on the defining representation of the group, is

$$\mathcal{M}(\omega) = e^{-i\omega Q/2} i\sigma_2 e^{i\omega Q/2}\ . \tag{62}$$

Because $\mathcal{M}$ anticommutes with Q, when a charged particle is transported around the vortex, its charge flips in sign. This monodromy property induces Aharonov-Bohm scattering of the charge eigenstates. Using the prescription Eq. (33), it is straightforward to compute

$$\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right)$$
$$= e^{-i\omega Q/2}\left(-i\sqrt{2}\right) e^{i\phi/2} \begin{pmatrix} \cos\phi/4 & -\sin\phi/4 \\ \sin\phi/4 & \cos\phi/4 \end{pmatrix} e^{i\omega Q/2}\ . \tag{63}$$

From Eq. (31), we thus obtain the cross section for scattering of charge eigenstates off of a fixed Alice vortex:

$$\sigma_\pm(\theta) = \frac{1}{2\pi p}\frac{1 \pm \sin\theta/2}{4\sin^2\theta/2}\ ; \tag{64}$$

here, σ_+ denotes the cross section when the scattered charge has the same sign as the original projectile, and σ_- is the cross section for charge-flip scattering. Note that these exclusive cross sections respect the relation Eq. (1) anticipated in the Introduction.

The case of a charged particle scattering from an Alice vortex is quite similar to the case of vortex-vortex scattering considered in Sec. V, where the orbit of the monodromy operator has order $n = 2$. There is an important difference, however—the monodromy operator Eq. (62) squares to -1 rather than 1. The property $\mathcal{M}^2 = -1$ holds whenever the charge of the projectile is odd, and hence the cross section Eq. (64) applies for any odd charge. The vanishing of σ_- in the backward direction is easily seen to be a consequence of $\mathcal{M}^2 = -1$; the trajectories with positive and negative *odd* winding numbers interfere destructively at $\theta = \pi$. If the charge of the projectile is even, then $\mathcal{M}^2 = 1$, and the cross section is given by Eq. (37) for $n = 2$, with $k = 0$ corresponding to σ_+ and $k=1$ to σ_-.

Now consider the case of vortex-vortex scattering, in the flux eigenstate basis. We denote by $|\omega\rangle$ the vortex state with flux $i\sigma_2 e^{i\omega Q}$. According to Eq. (7), the effect of an exchange on a state of two vortices, each with definite flux, can be expressed as

$$\mathcal{R}: |\omega_1, \omega_2\rangle \to |2\omega_1 - \omega_2, \omega_1\rangle \ . \tag{65}$$

The exchange preserves the "total flux" $i\sigma_2 e^{i\omega_1 Q} i\sigma_2 e^{i\omega_2 Q} = e^{i(\omega_2-\omega_1)Q} \equiv e^{i\omega_{\text{tot}}Q}$, and so an alternative notation is

$$\mathcal{R}: |\omega_1; \omega_{\text{tot}}\rangle \to |\omega_1 - \omega_{\text{tot}}; \omega_{\text{tot}}\rangle \ , \tag{66}$$

with the flux $\omega_2 = \omega_{\text{tot}} + \omega_1$ of the second vortex suppressed.

The two-vortex state can be decomposed into states with definite transformation properties under the centralizer of the total flux, which is U(1). These charge eigenstates also diagonalize the braid operator. The action of U(1) on the flux eigenstates is

$$e^{i\epsilon Q}: |\omega_1; \omega_{\text{tot}}\rangle \to |\omega_1 - 2\epsilon; \omega_{\text{tot}}\rangle \ , \tag{67}$$

and the charge eigenstates are

$$|q, \omega_{\text{tot}}\rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} d\omega' e^{iq\omega'} |2\omega'; \omega_{\text{tot}}\rangle \ , \tag{68}$$

where the charge q is an *even* integer. The braid operator acts on the charge eigenstates according to

$$\mathcal{R}: |q, \omega_{\text{tot}}\rangle \to e^{iq\omega_{\text{tot}}} |q, \omega_{\text{tot}}\rangle \ . \tag{69}$$

Formally, we can find the amplitude for a vortex with flux ω_1 to scatter from a fixed center with flux $\omega_2 = \omega_{\text{tot}} + \omega_1$ by applying Eq. (31). The result is

$$\langle \omega_1'; \omega_{\text{tot}} \ \text{out}| f(\phi) |\omega_1; \omega_{\text{tot}} \ \text{in}\rangle = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{1}{1+e^{i\phi}}\right) \frac{1}{\pi} \sum_q e^{iq(\omega'-\omega)} \left(e^{i(q\omega_{\text{tot}} - [q\omega_{\text{tot}}])(\phi/\pi - 1)} - e^{i(q\omega_{\text{tot}} - [q\omega_{\text{tot}}])(\phi/\pi + 1)}\right) \ , \tag{70}$$

where q is summed over even integers. We note that it is essential to subtract away the integer part of $q\omega_{\text{tot}}$ in order to obtain the correct result. For example, if ω_{tot} is rational, then the amplitude has support only for discrete values of $\omega'-\omega$. This would not have worked if the integer part had not been subtracted.

However, as noted above, this analysis is moot, because of the need to deal with the infrared divergent Coulomb energy of the states with $q \neq 0$. One way to screen the charge is to place another vortex pair far away, such that the four-vortex system carries total charge zero. But however we arrange to screen the charge, the state of the vortex pair we are studying will be correlated with the state of the compensating charge (unless the vortex pair is in a charge eigenstate). For example, our flux eigenstate becomes

$$|\omega_1; \omega_{\text{tot}}\rangle \to \frac{1}{\sqrt{\pi}} \sum_q e^{-iq\omega_1} |q; \omega_{\text{tot}}\rangle \otimes |-q; \text{screen}\rangle \ , \tag{71}$$

where $|-q; \text{screen}\rangle$ is the state of the screening charge. The vortex pair is actually in the mixed state

$$\rho = \frac{1}{\pi} \sum_q |q; \omega_{\text{tot}}\rangle\langle q; \omega_{\text{tot}}| \ . \tag{72}$$

The probability distribution for the scattered vortex will be the incoherent sum of the probability distributions for the braid eigenstates.

VIII. CONCLUSIONS

This paper has two recurring themes, relating to the non-Abelian Aharonov-Bohm effect and non-Abelian statistics. The first theme is that the non-Abelian Aharonov-Bohm effect provides a natural setting for multivalued physical observables. A particle that travels around a closed path returns to its starting point as a *different* kind of particle with different quantum numbers. This means that transition probabilities are not single-valued functions of the positions and quantum numbers of the particles in the final state. We have calculated cross sections that exhibit this multivalued character.

The second theme is that two particles that are "indistinguishable" need not be the same. The hallmark of non-Abelian statistics is that there can be an exchange contribution to an amplitude that interferes with the direct amplitude, even if the two particles that are exchanged are distinct objects with different quantum numbers. We have calculated cross sections that include such exchange effects.

These considerations illuminate some subtle aspects of non-Abelian gauge invariance. How do they relate to real phenomenology? There is no firm evidence that objects that obey non-Abelian statistics (called "nonabelions" in Ref. [40]) exist in nature. But it is surely conceivable that nonabelions will eventually be found, in strongly correlated electron systems [6, 40, 41], or other frustrated quantum many-body systems. An important question, then, is how would such objects be recognized in laboratory experiments? Much remains to be done to explore the many-body physics of nonabelions. Even the problem of three bodies is not very well understood.

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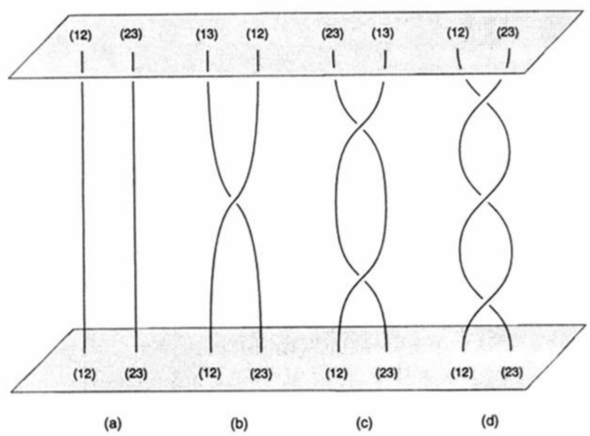

FIG. 6. Paths contributing to the amplitude for the propagation of a pair of vortices. The initial vortices carry flux taking the values (12) and (23) in S_3. If the vortices braid once as in (b) or twice as in (c), the quantum numbers of the pair are modified. But if the vortices braid three times as in (d), the final quantum numbers match the initial quantum numbers. Thus, paths (a) and (d) add coherently in the amplitude, although the two vortices change places.