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Internal Frame Dragging and a Global Analog of the Aharonov-Bohm Effect

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It is shown that the breakdown of a *global* symmetry group to a discrete subgroup can lead to analogs of the Aharonov-Bohm effect. At sufficiently low momentum transfer, the cross section for scattering of a particle with nontrivial Z_2 charge off a global vortex is almost equal to (but definitely different from) maximal Aharonov-Bohm scattering; the effect goes away at large momentum transfer. The scattering of a spin-$\frac{1}{2}$ particle off a magnetic vortex provides an amusing experimentally realizable example.

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The Aharonov-Bohm effect [1] is generally thought to be inextricably connected to gauge symmetry, and to quantum mechanics. However, upon reflection there are some funny aspects to these connections. When the flux Φ is expressed in terms of the fundamental unit h/e, so $\Phi \equiv \tilde{\Phi} h/e$, and the scattered charge is measured in terms of the fundamental unit e, so $q \equiv \tilde{q} e$, then the Aharonov-Bohm phase factor $\exp(iq\Phi/\hbar) = \exp(2\pi i \tilde{q}\tilde{\Phi})$ is independent of e and $\hbar$. This observation suggests that the Aharonov-Bohm effect might survive as e and $\hbar$ approach zero, if the limit is defined in a suitable way. We make these remarks not so much to outrage conventional wisdom concerning the Aharonov-Bohm effect, but to motivate the possibility of generalizing it. Can something like it occur for vortices of broken global symmetry, and in essentially classical contexts? We shall argue here that indeed it can, and that these generalizations have many potentially interesting incarnations.

(1) Frame dragging by broken symmetry.—To be definite let us consider a model with global U(1) symmetry broken down to Z_2 by condensation of a scalar field λ. Let η be a complex scalar field carrying half the U(1) charge of λ. Then generically one expects there to be a coupling of the type

$$\Delta\mathcal{L} = g\lambda\eta^2 + \text{H.c.} \tag{1.1}$$

In the homogeneous ground state where $\langle\lambda\rangle = v$ this term generates a mass splitting between the real and imaginary components of $\eta \equiv (\rho_1 + i\rho_2)/\sqrt{2}$:

$$\Delta\mathcal{L} \to \tfrac{1}{2}\Gamma(\rho_1^2 - \rho_2^2), \tag{1.2}$$

where $\Gamma \equiv 2gv$.

Now in a vortex configuration for λ, where $\langle\lambda(r,\phi)\rangle \to ve^{i\phi}$ outside a core region, it will still be possible to regard the interaction (1.1) as generating a mass splitting between two real fields. However, as ϕ varies the orientation of these fields in internal space is dragged along—in fact, it is rotated by $\phi/2$. In analyzing the dynamical effect of this frame dragging, it is convenient to work with fields which have a definite mass. Thus let us introduce the local mass eigenstates:

$$\tilde{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi/2} & e^{-i\phi/2} \\ -ie^{i\phi/2} & ie^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \eta \\ \eta^* \end{pmatrix}. \tag{1.3}$$

Because of the transformation (1.3) the wave equation will have two unusual features. (1) Each of the fields ρ_i obeys the boundary condition $\rho_i(\phi + 2\pi) = -\rho_i(\phi)$. This means that the allowed spectrum of partial waves includes only *half-odd integers.* (2) The gradient term $|\partial\eta|^2$ becomes modified, in its azimuthal component, to read $[(\partial_\phi + i\sigma_2/2)\tilde{\rho}]^2/2r^2$, where σ_2 is the Pauli matrix.

What is the effect of these modifications? At small momenta transfer $[4k^2\sin^2(\theta/2) \lesssim \Gamma]$ (where θ is the scattering angle) the second modification reduces to an additional potential

$$V_{A^2} = 1/4r^2. \tag{1.4}$$

Indeed ρ_1 and ρ_2 have different effective masses, and one should expect that perturbations connecting them are suppressed at small momentum transfer, enabling us to neglect the terms linear in σ_2. (This is not quite obvious, because the $1/r^2$ interaction is potentially singular. However, here the fact that the allowed partial waves are half integral saves the day, because it means that there is always a centrifugal barrier shielding the origin.) Thus the only significant effect of the interaction with the vortex is to modify the boundary conditions, and to add an additional potential (1.4). We shall compute the resulting cross section, and justify our neglect of the off-diagonal terms, in the following section. If we neglected the additional potential (1.4), then we would have exactly the setup which leads to maximal Aharonov-Bohm scattering. The additional term introduces a calculable modification, which is relatively small for high partial waves (or small angles).

On the other hand, clearly as $\Gamma \to 0$ the effect of the vortex must go away (for all finite angles $\theta > \sqrt{\Gamma}/k$), apart from a possible contribution from ordinary scattering off the core (in the lowest partial wave). Thus at large momentum transfer $4k^2\sin^2(\theta/2) \gg \Gamma$ the induced gauge field must essentially cancel the effect of the modified boundary conditions. Notice that the induced "gauge field" appearing in the gradient energy, far from being responsible for the Aharonov-Bohm-like scattering, plays a crucial role in canceling it off.

We may think of the masses of ρ_1 and ρ_2 as the eigenvalues of the Hamiltonian of a two-level system, where the Hamiltonian depends on an external parameter, the angle around the vortex. Then we recognize that the minus sign in the boundary condition satisfied by the mass eigenstate fields on the vortex background is an instance of Berry's phase [2]. Indeed this point of view is instructive on several counts. The restriction to low momenta we found above may be considered as the adiabatic condition for applicability of Berry's phase. Also the special role of the vortex topology is clarified— in circling the core, we surround a point where an irremovable degeneracy between the masses of ρ_1 and ρ_2 occurs. The induced gauge connection that arises when the Lagrangian is expressed in terms of the mass eigenstate fields is precisely Berry's connection. It is noteworthy that, although this connection is purely off diagonal in the mass eigenstate basis, it has a nontrivial effect on the dynamics in the adiabatic limit, because its square is an on-diagonal scalar potential. This induced diagonal potential arises quite generally in the adiabatic approximation for systems in which "light" and "heavy" degrees of freedom are coupled together.

If the U(1) broken symmetry *were* a gauge symmetry, then the gauge field induced by the transformation (1.3) would be exactly canceled by the true gauge field present in the gauge covariant derivative of η in the vortex background. Then we would have the classic Aharonov-Bohm scattering induced by the change in boundary conditions, at all momenta. Related to this, in a broken gauge theory the scattering described here, which (since it arises from the coupling to the scalar Higgs field) might appear to be additional to the classical Aharonov-Bohm scattering, in a sense reduces to an alternative representation of it.

Thus far we have considered the case of Z_2 charges. For higher global charges, a more complex situation emerges. Consider for concreteness a Z_3 charge. The interaction corresponding to (1.1) is

$$\Delta\mathcal{L} = g\lambda\eta^3 + \text{H.c.} \tag{1.5}$$

The equation of motion for η receives a contribution of order η^2 from (1.5). Thus, for small amplitudes its effect is negligible. In particular, there is no scattering from the λ vortex, even for small momenta, in the low amplitude limit. On the other hand, for finite amplitude waves an analysis similar to the one given above applies. For small momenta (where "small" now depends on the amplitude of the wave) it will be appropriate to diagonalize (1.5), and one will find the appropriate Aharonov-Bohm-like cross section.

One might be concerned that, since the Nambu-Goldstone excitations associated with the broken symmetry field λ are massless and therefore may be radiated with arbitrarily little energy, the effect discussed here could be washed out by Nambu-Goldstone boson emission. However, since the Nambu-Goldstone field is derivatively coupled, it is clear that on general grounds its emission is an order $(k/F)^2$ correction to the elastic process for small momenta, where F is the scale of symmetry breaking. Therefore it can be made arbitrarily small in regimes of interest, and clearly cannot wash out the generic effect discussed here.

An essentially geometrical cross section associated with a global symmetry poses a potential paradox; it is noteworthy how this paradox is resolved. While gauge charges have a universal coupling strength, global charges do not, and so it is difficult at first hearing to understand how an essentially geometrical, parameter-independent form of the cross section could emerge for them. What we have seen is that there is a geometrical cross section determined by the global charge, but the *domain of validity* of the cross section, i.e., the range in momenta and angle for which it is valid, is a nonuniversal parameter that depends on the strength of the allowed coupling that fixes the charge assignment. As the strength of this coupling goes to zero (removing, in principle, our ability to define the charge) the form of the cross section remains unchanged where it is valid, but its range of validity shrinks to zero.

(2) Calculation.— Now we shall treat the prototype problem discussed above more quantitatively. For simplicity we will treat the nonrelativistic case. We will also consider the quantum-mechanical scattering problem, although similar considerations would apply to the classical scattering waves.

The substitution of Eq. (1.3) into the equation for the

(η,η^*) modes results in a nonrelativistic Schrödinger equation of the form

$$i\partial_t \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix} -\frac{1}{2\mu_1}\left(\nabla^2-\frac{1}{4r^2}\right)+\mu_1 & -\frac{\partial_\phi}{2\mu_1 r^2} \\ \frac{\partial_\phi}{2\mu_2 r^2} & -\frac{1}{2\mu_2}\left(\nabla^2-\frac{1}{4r^2}\right)+\mu_2 \end{pmatrix} \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} \tag{2.1}$$

together with the boundary condition $\rho_i(\phi+2\pi) = -\rho_i(\phi)$, and where the perturbed masses are $\mu_{(1,2)}^2 = (m^2 \pm \Gamma)^{1/2}$. The off-diagonal entries (terms linear in the induced effective gauge field) connect states of different effective mass. Therefore, at low incident momenta it is reasonable to expect that their effect will be small. Our strategy will be to first solve the scattering problem ignoring the off-diagonal terms, and then take them into account perturbatively. Of course, if we send in a pure ρ_2 state then there can be no real ρ_1 production for incident momenta below the threshold enforced by energy conservation. However, even below this threshold, the off-diagonal terms can affect the elastic scattering of the ρ_2 modes at second order in perturbation theory. We will argue below that this effect is indeed small for incident energies much less than the mass splitting.

The solution of the diagonal scattering problem proceeds by performing a mode expansion (in two spatial dimensions—or in three at normal incidence)

$$\rho_2(t,r,\phi) = \sum_{n\in Z} e^{-i(\omega+\mu_2)t} e^{i(n+1/2)\phi} P_n^{(2)}(r) \tag{2.2}$$

(similar for ρ_1). Note that the partial-wave expansion is shifted by one-half due to the boundary conditions. Defining $z=k_2 r$ where $\omega=(k_2)^2/2\mu_2$ we find that the radial eigenfunctions $P_n(z)$ satisfy a Bessel equation of order $\nu_n^2=(n+\frac{1}{2})^2+\frac{1}{4}$. The shift of $\frac{1}{4}$ from the usual order, $\nu_n^2=(n+\frac{1}{2})^2$, expected with a mode expansion of the form Eq. (2.2), is due to the on-diagonal induced potential, Eq. (1.4).

To select the appropriate set of solutions we must demand self-adjointness of the Hamiltonian, and square integrability of the solution at the position of the vortex. However, as discussed in the appendix of [3] (in the context of gauge strings), this still leaves a one-parameter family of allowed boundary conditions in both the $n=-1$ and $n=0$ modes. The correct choice is discovered by first performing a calculation at finite core radius R, and then taking the limit $R\to 0$ [3]. The result is that we should use only positive-order Bessel functions in *all* modes.

Now we are ready to construct the scattering solution and calculate the elastic differential cross section for incident ρ_2 modes. This is most easily done if we reexpress the selected Bessel functions in terms of outgoing $(H_{\nu_n}^{(1)})$ and incoming $(H_{\nu_n}^{(2)})$ Hankel functions. If we take the incident wave to be a plane wave $\exp(-ik_2 x)$ then we must construct out of the Hankel functions a solution of the form

$$\rho_2^{\rm sol} = \frac{1}{2}\sum_{n\in Z} e^{i(n+1/2)\phi} e^{-i\pi\alpha_n/2} \times [H_{\nu_n}^{(2)}(k_2 r) + H_{\nu_n}^{(1)}(k_2 r)] , \tag{2.3}$$

where ν_n is given by the positive square root. The α_n are determined by demanding that Eq. (2.3) match onto the incoming plane wave plus an outgoing scattered wave at infinity; we require

$$\rho_2^{\rm sol} \sim e^{i\phi/2}[e^{-ik_2 x} + f(\phi) e^{ik_2 r}/\sqrt{r}] , \tag{2.4}$$

where $f(\phi)$ is the scattering amplitude. The phase $e^{i\phi/2}$ in front is necessary, because of the double valuedness of our solution, but it is harmless—if we construct narrow wave packets that travel in along the positive x axis, then this phase is trivial. Using the usual expansion of the plane wave in terms of integer-order Bessel functions

$$\exp(-ikr\cos\phi) = \sum_{n\in Z} e^{-i\pi|n|/2} e^{in\phi} J_{|n|}(kr) \tag{2.5}$$

and the asymptotic behavior of the Hankel functions, the constraint of matching onto the plane wave determines $\alpha_n=\nu_n$ for all n.

We can now calculate the phase shifts $\delta_n(k_2)$ defined by the asymptotic relation

$$\rho_2^{\rm sol} \sim \frac{1}{2}\sum_{n\in Z} e^{i(n+1/2)\phi} e^{-i\pi|n|/2} \times [H_{|n|}^{(2)}(k_2 r) + e^{i\delta_n(k_2)} H_{|n|}^{(1)}(k_2 r)] . \tag{2.6}$$

A simple calculation involving the asymptotic behavior of the Hankel functions leads to the result $\delta_n = \pi(|n|-\nu_n)$. From these phase shifts we can calculate the scattering amplitude $f(\phi)$. The result is

$$f(\phi) = \frac{e^{-i\phi/2}}{(2\pi i k_2)^{1/2}} \left[\frac{1}{\cos(\phi/2)} + 2\sum_{n=0}^{\infty} (-1)^n (e^{i\Delta_n}-1)\cos[(n+\tfrac{1}{2})\phi] \right] , \tag{2.7}$$

where $\Delta_n = \pi\{n+\frac{1}{2} - [(n+\frac{1}{2})^2+\frac{1}{4}]^{1/2}\}$. The first term in Eq. (2.7) is the usual maximal Aharonov-Bohm amplitude. The corrections are due to the diagonal $1/4r^2$ potential, and are largest in low partial waves.

The differential scattering cross section, expressed in terms of the scattering angle $\theta=\pi-\phi$, has the form

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k_2}\frac{1}{\sin^2(\theta/2)}[1+C(\theta)] ; \tag{2.8}$$

it is the maximal Aharonov-Bohm cross section times a calculable correction factor that approaches 1 at small angles. The function $C(\theta)$ is found by numerically evaluating the sum in Eq. (2.7), which actually converges quite slowly. The correction is largest ($C=0.202$) at $\theta=\pi$.

A calculation in second-order perturbation theory shows that the effect of the neglected off-diagonal terms on the elastic scattering of ρ_2 is bounded by a constant times $(k_2)^4/\Gamma^2$, uniformly in all partial waves [4].

(3) Examples.— (a) Spin-$\frac{1}{2}$ scattering by a magnetic vortex. Consider a material with a planar magnetization $\mathbf{M}(x)$—namely, a material described by the XY model. This model, of course, supports vortices, which indeed play an important role in its dynamics. A spin-$\frac{1}{2}$ particle, which might be an electron or a neutron, for example, couples to the magnetization with an interaction

$$\Delta\mathcal{H}=g\psi^\dagger\sigma\psi\cdot\mathbf{M}\,, \tag{3.1}$$

where ψ is the spinor field representing the particle. The scattering of the spin-$\frac{1}{2}$ particle from the magnetic vortex is an instance of the general analysis above, but let us state it in fresh terms. In the presence of a vortex, the frame of the spin is dragged around. Thus if the magnetization is given by the vortex form $\mathbf{M}^i(r,\phi)\to M_0(-\delta_{i1}\times\sin\phi+\delta_{i2}\cos\phi)$, then to keep the effective-mass term generated by the interaction (3.1) diagonal, we shall need to transform to the frame-dragged variable $\tilde\psi\equiv\exp(i\phi\sigma_3/2)\psi$. Now as a spinor is rotated through 2π, its sign changes. Thus, for consistency, at low momentum where parallel transport of the spin is appropriate, the boundary condition on the spinor wave function requires it contain only half-odd-integer angular momenta. As there also occurs an induced diagonal potential (1.4), the spin will scatter off the vortex with the cross section (2.8).

Various generalizations may be considered. For example if the magnetization is tipped out of the plane by angle β and sweeps out a cone, then the calculation of the cross section changes as follows. Upon diagonalizing the interaction (3.1) we find that the effective mass term takes the form $\tilde\psi^\dagger(\sin\beta\,\sigma_3+\cos\beta\,\sigma_2)\tilde\psi$, with eigenspinors $\tilde\psi_\pm=e^{-i(\beta/2)\sigma_1}(1,\pm i)^T$. Now between *these* eigenspinors the effective gauge potential proportional to σ_3 in the modified gradient term $|\partial_\phi\psi|^2=|(\partial_\phi-i\sigma_3/2)\tilde\psi|^2$ *does* not have nonvanishing diagonal matrix elements, which must be included in the calculation. As a result, the quantities v_n are modified to become

$$v_n^2=(n+\tfrac{1}{2})(n+\tfrac{1}{2}\pm\sin\beta)+\tfrac{1}{4}\,, \tag{3.2}$$

where the $\pm$ refers to the different eigenspinors. From these the cross section is readily computed, but the formula is not particularly transparent. It is noteworthy, however, that the leading correction to the canonical Aharonov-Bohm result contains terms in $\sin\phi$ as well as $\cos\phi$, giving explicit parity and time-reversal asymmetries.

(b) Polarized light. The essential requirement for the analysis of the previous section to apply is that there should be 2 degrees of freedom with different dispersion relations that are rotated into one another by the variation of a material parameter, such as a magnetization, and that when the material parameter rotates through a closed cycle each degree of freedom returns to itself, with a change of phase. This general setup can be realized in a variety of optical contexts, where the degrees of freedom are two polarizations of light of a given frequency. Realizations of frame dragging for polarized light have already been used for interference experiments [5]; we are merely adapting it to a realization in scattering. Of course, there is nothing special about the optical region of the electromagnetic spectrum in this regard, and an alternative macroscopic realization could be constructed for microwaves propagating through ferrites.

(c) Passport to exotica. Quite a few remarkable phenomena involving among others Alice strings [6], Cheshire charge [7], and flux-tube–flux-tube scattering [8] have been studied in the context of spontaneously broken non-Abelian gauge symmetries. Unfortunately, however, the list of spontaneously broken non-Abelian gauge symmetries available for experimental manipulation is vanishingly small. The main point emphasized above, that frame-dragging phenomena usually associated with gauge theories also occur at low momenta for broken global symmetries, opens the strong possibility that effects closely analogous to these can be realized in suitable laboratory condensed-matter systems. Particularly interesting in this regard are helium 3 [9] and liquid crystals, which are known to have complicated order-parameter spaces and to support non-Abelian vortices [10].

The existence of an infinite range Aharonov-Bohm interaction between a global Z_2 charge and a global string suggests that the Z_2 charge on a black hole should be measurable [11]. However, we have seen that Aharonov-Bohm scattering occurs only if the passage of the charge by the string can be regarded as adiabatic. We believe that gravitational time dilation makes it impossible to satisfy this criterion as the charge falls into a black hole, so that the long-range interaction between charge and string is destroyed [12].

The effects we have described are important for the interaction of matter with global and axion strings, and may affect their evolution in the early Universe.

These matters are under active investigation [4].

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