2. Field Theory on Curved Spacetime

Our experience with field theory on flat spacetime has prepared us to confront the problem of constructing quantum fields on a curved background.

The idea is the idea that we always invoke to promote flat-spacetime physics to general covariant physics. Locally (at sufficiently short distances) spacetime is approximately flat. Our field theory on curved spacetime should reduce to our flat-spacetime theory locally. Because we must consider local physics, it is essential that we describe the quantum field theory in terms of local field variables.

If our flat spacetime theory is causal, then so will the theory on curved spacetime be causal — if it reduces to the flat theory locally. If information does not propagate outside the light cone locally, then it stays within the light cone globally.

The concept of a particle, which was fundamental in our discussion of flat spacetime, is less essential in the formulation of field theory on a curved background. A particle, as we defined it, is an IR of the Poincare group. But a curved background will not be Poincare invariant, nor will the quantum theory built on it admit a unitary rep. of the Poincare group. Therefore
of a particle as an approximate one, valid when the wavelength is much smaller than the length scale characteristic of the curvature. (In practice, this limitation need not be serious. E.g., the width of the $Z^0$ is not very sensitive to the Hubble constant $H_0$.)

To begin, we construct the Klein-Gordon classical field theory on a nontrivial background. The flat space action

$$S = \int d^4x \sqrt{g} \left[ \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$$

may be written

$$S = \int d^4x \sqrt{g} \quad \frac{1}{2} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$$

where $g = -\det(g_{\mu\nu})$, so that $d^4x \sqrt{g}$ is the invariant volume element. In this form, $S$ is invariant under general coordinate transformations

$$x \rightarrow x'(x)$$

where $\phi$ is a scalar transforming as

$$\phi(x) \rightarrow \phi(x')$$

From this action, we derive the Euler-Lagrange equation ---
\[ \partial_\mu \partial^\mu \phi - \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

where
\[ Z = \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) \]

or
\[ \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \ g^{\mu\nu} \partial_\nu \phi) + m^2 \phi = 0 \]

We can put this equation in a more recognizable form by invoking an identity:
\[ \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} = \Gamma^\lambda_\mu \nu \]

Thus, we have
\[ \partial_\mu \partial^\mu \phi + \Gamma^\lambda_\mu \nu \partial_\mu \phi + m^2 \phi = 0 \]

or
\[ (\nabla_\mu \nabla^\mu + m^2) \phi = 0 \]

where the covariant derivative \( \nabla_\mu \phi \) of a scalar is
\[ \nabla_\mu \phi = \partial_\mu \phi \]

and if a 4-vector is
\[ \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_\mu \alpha V^\alpha \]

\[ \Rightarrow \nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\nu_\mu \nu V^\nu \]

is covariant divergence.

It is the flat space KG eqn, with derivative replaced by covariant derivative.
To derive the identity, recall that for any matrix \( M \)

\[
\ln \det (M+SM) = \ln \det (M+SM) = \ln M + \ln (I+M^{-1}SM) = \ln M + \text{Tr} M^{-1}SM
\]

or

\[
\frac{S(\det M)}{\det M} = \frac{1}{2} \text{Tr} M^{-1}SM
\]

Thus

\[
S \sqrt{g} = \frac{1}{2} \sqrt{g} \, g^{\mu \nu} S g_{\mu \nu}
\]

Therefore

\[
\frac{1}{\sqrt{g}} \partial\mu \sqrt{g} = \frac{1}{2} g^{\mu \nu} \partial\mu g_{\nu \nu}
\]

Recalling

\[
\Gamma_{\mu \nu}^\lambda = \frac{1}{2} g^{\lambda \sigma} \left[ \partial_\mu g_{\nu \sigma} + \partial_\nu g_{\mu \sigma} - \partial_\sigma g_{\mu \nu} \right]
\]

\[
\Rightarrow \Gamma_{\lambda \mu}^\nu = \frac{1}{2} g^{\lambda \sigma} \left[ \partial_\sigma g_{\mu \nu} + \partial_\nu g_{\lambda \sigma} - \partial_\mu g_{\lambda \nu} \right]
\]

\[
= \frac{1}{2} g^{\lambda \sigma} \partial_\sigma g_{\lambda \nu}
\]

we have

\[
\frac{1}{\sqrt{g}} \partial\mu \sqrt{g} = \Gamma_{\lambda \mu}^\nu
\]
It is convenient to formulate the field theory in terms of an action principle, because then we can easily extract the stress tensor that acts as a source in the Einstein equation.

If the action for gravity coupled to matter is

\[ S = S_{\text{grav}} + S_{\text{matter}} \]

where

\[ S_{\text{grav}} = \frac{-1}{16\pi G} \int d^4x \sqrt{|g|} R \]

and we vary the metric

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \]

then

\[ \delta S_{\text{grav}} = \frac{-1}{16\pi G} \int d^4x \sqrt{|g|} G^{\mu\nu} \delta g_{\mu\nu} \]

where

\[ G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \]

We may now define a stress tensor by

\[ \delta S_{\text{matter}} = -\frac{1}{2} \int d^4x \sqrt{|g|} T^{\mu\nu} \delta g_{\mu\nu} ; \]

then the eqn of motion is

\[ G^{\mu\nu} = -8\pi G T^{\mu\nu} \]

– the Einstein equation
To derive $T^\mu_\nu$ from

$$S = \int \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$$

we use

$$\sqrt{g} = \frac{1}{2} \sqrt{g} \; g^{\mu\nu} g_{\mu\nu}$$

and

$$g^{\mu\nu} = -g^{\sigma\rho} g_{\nu\rho} g_{\mu\sigma}$$

which follows from $gM^{-1} = -M^{-1}SM$.

Thus

$$T^\mu_\nu = \partial^\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \left[ g^{\sigma\rho} \partial_\sigma \phi \partial_\rho \phi - m^2 \phi^2 \right]$$

Note: general word: invariance of Smatrix $\Rightarrow \nabla \cdot T^\mu_\nu = 0$

Now we want to construct a Hilbert space such that the fields acting on this space are conserved. To do this, we will make use of the solutions to the classical KG eqn. in which a natural "inner product" can be defined.

Let $f, g$ be solutions to

$$(\nabla_\mu \nabla^\mu + m^2) f(x) = 0$$

The Klein-Gordon "inner product" of two solutions is defined by choosing a spacelike surface $\Sigma$, and integrating ---
\[(f, g) = i \int d^3 x \sqrt{h} \, n^m \left[ f^* \partial_m g - (\partial_m f^*) g \right] \]

where \( h \) is the induced 3-metric on the surface \( \Sigma \) and \( n^m \) is the forward-pointing unit normal to \( \Sigma \). This form has the desirable property of being independent of the slice \( \Sigma \). This follows from Gauss's Theorem, which can be written in the form

\[
\int d^4 x \sqrt{g} \, \nabla^k V_m = \int d^3 x \sqrt{h} \, n^k V_m \]

where \( V_m \) is a 4-vector.

( Remember that \( \nabla^k V_m = \frac{1}{\sqrt{g}} \partial^k (\sqrt{g} V_m) \)

and that \( g \) reduces to \( h \) in an orthonormal coordinate system with \( n^m = (1, 0, 0, 0) \).

Hence

\[
(f, g)_{\Sigma_1} - (f, g)_{\Sigma_2} = \int d^3 x \sqrt{h} \, n^m \left( f^* \partial_m g - \partial_m f^* g \right) \bigg|_{\Sigma_1, \Sigma_2} 
\]

\[
= \int d^4 x \sqrt{g} \, \nabla^k \left( f^* \partial_m g - \partial_m f^* g \right) \bigg|_{\Sigma} 
\]

(where \( \Sigma = \Sigma_1 - \Sigma_2 \))

= 0 if \( f \) and \( g \) solve Klein-Gordon

just as in the flat space case.
Remarks

- It is of course implicit in this construction that the Klein-Gordon equation has solutions that are globally defined in the spacetime. For this purpose, it suffices that the spacetime be "globally hyperbolic" — that is, not just have a Cauclay Surface.

What is a "Cauchy surface"? It is first of all a "closed achronal slice" of the spacetime — a 3-surface $\Sigma$ such that no two points on $\Sigma$ are connected by a time-like or null curve. To be Cauchy, $\Sigma$ must also have the following property:

For every point $p$ in the spacetime, every time-like or null curve through $p$ (without a past or future endpoint) must cross $\Sigma$.

The idea of this definition is that the physics on $\Sigma$ completely determines the physics in the future and past of $\Sigma$. So, in particular, initial data on $\Sigma$ is sufficient to determine the solution to the Klein-Gordon equation throughout the spacetime.

In fact, it turns out that if the spacetime has one Cauchy surface, then there is a Cauchy surface through every point. Furthermore, we can choose
a time coordinate $t$ so that each $t$-constant surface is Cauchy. (Technically, a time coordinate is a scalar function $t(x)$, assumed smooth such that $\partial^2 t / \partial x^2 > 0$.)

For globally hyperbolic spacetimes, Ken, show with this space the property that

\[
\{ \text{global solutions to KG eqn} \} = \{ \text{initial data on a "time slice" $\Sigma$} \}
\]

References

R. Wald, "General Relativity", Chapter 8, 10

The existence of a Klein-Gordon inner product that does not depend on the slice $\Sigma$ actually holds more generally for a free field on a curved background, it applies to any coupling to external sources such that the field eqn is linear and homogeneous (action $S$ is quadratic). E.g. it applies to Klein-Gordon field coupled to external electromagnetic field.
Even restricting to a free scalar field on a non-rotational geometry, the action that we constructed is not the most general one that is quadratic in \( \phi \). For example, we could have included in the action the term

\[
S' = \int d^4x \sqrt{-g} \left( -\frac{1}{2} \mathcal{R} \phi^2 \right)
\]

where \( \mathcal{R} \) is the scalar curvature. Then the field eqn would become

\[
\left( \nabla^2 \phi + m^2 + \mathcal{R} \phi \right) = 0
\]

More complicated terms in the field eqn involving other invariants constructed from the curvature would also be included (but would be expected to be suppressed by powers of \( \mathcal{R} \) on \( m^2 \) - Planck - and so would be negligible for curvature small in Planck units). For any such field eqn, we can construct a K-G inner product. But for now we will ignore any such additional dependence on the BG and continue to consider equation

\[
\left( \nabla^2 \phi + m^2 \right) \phi = 0
\]

which already serves to illustrate some of the features of QFT on a non-rotational background.
In constructing a quantum theory, we will initially ignore the "back reaction" of the field \( \phi \) on the geometry, expanded by \( \delta \mu = -8\pi G \mathcal{T}^{\mu \nu} \). We will consider the geometry to be a classical source that is not influenced by the (quantum) field \( \phi \). Later, we will attempt to discuss some back reaction effects.

On a globally hyperbolic spacetime, which has a well-behaved, globally defined time coordinate, we can perform canonical quantization of the classical free Klein-Gordon theory. We choose time slices and then impose

\[
\begin{align*}
\{ \phi(x), \phi(y) \} \ll_{t.e.} &= i \delta^3(x-y) \\
\{ \phi(x), \phi(y) \} \ll_{t.e.} &= 0 = \{ \phi(x), \phi(y) \}
\end{align*}
\]

We may put this in a more covariant looking form by denoting a Cauchy surface by \( \Sigma \), and letting \( \nu^\mu \) be the normalized forward-pointing normal to \( \Sigma \). Then

\[
\begin{align*}
\{ \phi(x), \nu^\mu \partial_\mu \phi(y) \} \ll_{\Sigma} &= i \frac{1}{\sqrt{\delta}} \delta^3(x-y) \\
\{ \phi(x), \phi(y) \} \ll_{\Sigma} &= 0 \\
\{ \nu^\mu \partial_\mu \phi(x), \nu^\mu \partial_\mu \phi(y) \} \ll_{\Sigma} &= 0
\end{align*}
\]
Here all commutators are evaluated for two fields on the slice $\Sigma$; has the induced 3-metric on $\Sigma$, and $\frac{1}{\sqrt{h}}$ is the appropriate $f$ function normalization for integration against $\text{co} \text{invariant induced volume element } \sqrt{h}$.)

In fact, if we impose the canonical commutation relations (CCR) on any spacialike $\Sigma$, then they are automatically satisfied on every other spacialike slice, provided that $\phi(x)$ satisfies the $K\phi$ eqn. To see this, we first show:

- $\phi$ satisfies CCR on $\Sigma$ if and only if
  $$\left[(f, \phi)_{\Sigma}, (g, \phi)_{\Sigma}\right] = -(f, g^*)_{\Sigma}$$
  for any two solutions $f, g$ to the $K\phi$ eqn.

Thus, if $\phi$ satisfies the $K\phi$ equation, we may use the slice independence of $(\cdot, \cdot)_{\Sigma}$ to show.

- $\phi$ satisfies CCR on $\Sigma \iff \phi$ satisfies CCR on $\Sigma'$, where $\Sigma$ and $\Sigma'$ are any two spacialike slices.

To show the only if part of the first statement above, recall

$$\left(f, g\right)_{\Sigma} = \int d^3x \sqrt{h} \ n^\mu \left(f^* \partial_\mu g - \partial_\mu f^* g \right)$$
and evaluate
\[
[(f, \phi)_\Sigma, (g, \phi)_\Sigma]
\]
\[
= \left[ i \int d^3x \sqrt{\Sigma} n^\mu \left( f^\mu \partial_\mu \phi - \partial \phi \right),
\right.
\]
\[
\left. i \int d^3x' \sqrt{\Sigma} \ T^\nu \left( g^\nu \partial_\nu \phi - \partial \phi \right) \right] \Sigma
\]
\[
= - \int d^3x \sqrt{\Sigma} \int d^3x' \sqrt{\Sigma} \left[ f^\mu \partial_\mu \phi \right]_{\Sigma} \left[ \left( g^\nu \partial_\nu \phi \right)_{\Sigma} \right]
\]
\[
= i \int d^3x \sqrt{\Sigma} \left( 2 \partial_x f^\mu (\phi) \text{g}^\nu (x) - f^\mu (x) \partial_x \text{g}^\nu (x) \right)
\]
\[
= - (f, g^x)
\]

Thus, for a field that satisfies the KG equation, we find that

(i) If the ccr are imposed on any spacelike slice, they hold on all slices. The quantum theory that we construct does not depend on how we slice the spacetime.

(ii) The theory is causal in the sense that 
\[
[(\phi (x), \phi (y)) = 0
\]

whenever there is a spacelike slice that contains both x and y. (For globally hyperbolic spacetimes, this slice presumably exists if there are no timelike or null curves connecting x and y.)
Now we wish to proceed with the construction of the "Fock space", the Hilbert space of this theory. As was emphasized in the discussion of the flat space theory, the construction of the Fock space does not require that the notion of a particle apply. We need only be able to divide the space of solutions of the KG eqn into subspaces with pos. def. and neg. def. KG norms.

The existence of a complete basis for the solutions to the KG eqn such that the inner product on $\Sigma$ satisfies

$$(\phi_i, \phi_j)_\Sigma = \delta_{ij}$$
$$(\phi_i, \phi_j^*)_\Sigma = 0$$
$$(\phi_i^*, \phi_j^*)_\Sigma = -\delta_{ij}$$

follows from the property that initial data on $\Sigma$ determines a unique solution, for arbitrary initial data can be expanded in such a basis. (Note that we are now using a schematic notation in which discrete normalization of the solutions is assumed.) This is particularly easy to see in the case where the slice $\Sigma$ has the topology of $\mathbb{R}^3$. In that case, we can smoothly deform the geometry of $\Sigma$ into the sufficiently distant part on $\Sigma$.\[\Sigma\] Flat
Then we may choose such a frame in the flat region and propagate it about to $\Sigma$ to get the desired basis in the vicinity of $\Sigma$. (The $\mathbb{K}$ inner product remains independent, and depends only on the solution and its first derivative at the slice.)

Now if we expand the field $\phi$ in terms of this basis

$$\phi = \sum_i (u_i a_i + u_i^* a_i^+)$$

we have

$$\langle u_i, \phi \rangle = a_i$$

$$-\langle u_i^* , \phi \rangle = a_i^+$$

(evaluated on any slice)

Therefore, the CCR in the form

$$[ (f, \phi), (g, \phi) ] = -(f, g^*)$$

imply

$$\langle a_i, a_j \rangle = -\langle u_i, u_j^* \rangle = 0$$

$$\langle a_i^+, a_j^+ \rangle = 0$$

$$\langle a_i, a_j^+ \rangle = -(u_i, -u_j) = \delta_{ij}$$

That is, the $a_i, a_i^+$ are conventionally normalized creation and annihilation operators.

And since $[ (f, \phi), (g, \phi) ] = -(f, g^*)$ for a complete basis of solutions implies the CCR in any spacelike $\Sigma$, we could just as well assume $\mathbb{K}$ at commutators and then inter the canonical commutators.
that is, an alternative to canonical quantization is to choose a complete basis $\{u, u^*\}$ of solutions to $\mathcal{K}G$, and construct $\mathcal{H}''$ - the Hilbert space spanned by $\mathcal{K} \{u, u^*\}$. Then extend to the full space $\mathcal{H}$ and define $\mathcal{A}$ on $\mathcal{H}$.

Finally, we construct the field $\phi$ on $\mathcal{H}$, which we have shown is local - i.e., satisfies $[\phi(x), \phi(y)] = 0$ for all $x$ and $y$ from a spacelike slice.

In the construction of the full space $\mathcal{H}$, there is, however, an ambiguity. There are many ways to choose the basis $\{u, u^*\}$ of solutions with positive Klein-Gordon norm. E.g., if $u$ satisfies

$$(u, u) = 1 \quad (u, u^*) = 0 \quad (u^*, u^*) = 1$$

then

$u^* = \cosh \Theta u + \sinh \Theta u^*$

$satisfies$

$$(u^*, u^*) = \cosh^2 \Theta - \sinh^2 \Theta = 1$$

$$(u^*, u^*) = \cosh \Theta \sinh \Theta - \sinh \Theta \cosh \Theta = 0$$

A linear combination of pos and neg norm solution is just as acceptable as $u$.

There is a natural way to decompose the space of solutions to the KG equation into subspaces on which the KG inner product is pos, def, and neg, def, respectively, only in the special case if
a stationary spacetime (like flat space). We say that the spacetime is stationary if the time coordinate $t$ can be chosen so that the metric is $t$-independent. Then $\frac{\partial}{\partial t}$ generates a symmetry of the geometry, and is said to be a timelike "Killing vector." (The spacetime is invariant under time translations.)

Since time translation is a symmetry, if $u(t, x)$ is a solution to the KG equation, then so is

$$u(t + \Delta t, x) = u(t, x) + \Delta t \frac{\partial}{\partial t} u(t, x).$$

Therefore, the solutions transform as a representation of the time translation group, and we may decompose this representation into (one-dimensional) irreducible representations. In other words, the operator

$$H = i \frac{\partial}{\partial t}$$

preserves the space of solutions, and we can diagonalize $H$ on this space:

$$H u_k = \omega_k u_k \quad \text{where } \omega_k \text{ is the frequency of the solution}. $$

The solutions have the form

$$u_k(t, x) = e^{-i \omega_k t} \phi_k(x).$$
From \((f,g) = \int d^3 \sqrt{\det} \left( f \dot{g} - \dot{f} \cdot \nabla g \right)\), we see that, since the inner product is independent of \(t\), we have for \(\dot{\det} = 0\),

\[
(f(t), g(t)) = (f(t+dt), g(t+dt))
\]

\[
= (f, g)_t + dt[(f, \dot{g}) + (f, \ddot{g})]
\]

\[
\Rightarrow (\dot{f}, g) + (f, \ddot{g}) = 0
\]

So for solutions of definite frequency \(\omega_k = \pm \omega\), we have

\[
i(\omega_k - \omega_j)(u_k, u_j) = 0
\]

\[
\Rightarrow \text{solution of distinct frequency are orthogonal in the KB inner product}
\]

Since the inner product

\[
(u_k, u_j)_{t=0} = (\omega_k + \omega_j) \int d^3 x \, u_k^*(x) u_j(x)
\]

is evidently positive for positive frequency solutions, we have found a basis the working basis) such that

\[
(u_i, u_j) = \delta_{ij}
\]

\[
(u_i, \dot{u}_j) = 0
\]

\[
(u_i, \ddot{u}_j) = -\delta_{ij}
\]

Acting on this basis, the operator \(H\) is non-negative. If we define operators \(a_i\) and a state \(\mid 0 \rangle\) such that

\[
a_i \mid 0 \rangle = 0\]

\[
\mid i \rangle = a_i^\dagger \mid 0 \rangle \quad \text{where} \quad \langle i | j \rangle = \delta_{ij}
\]
we have \[ [a_i, a_j^+] = \delta_{ij} \quad [a_i, a_j] = 0 \quad [a_i^+, a_j^+] = 0 \]

acting on Fock space \[ \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} \]

and the Hamiltonian \( \mathcal{H} \) has the representation
\[ \mathcal{H} = \sum_k \omega_k \, a_k^+ a_k \]

But in general, in the case of a nonstationary spacetime, there is no natural choice for the positive norm subspace \( \mathcal{H} \) of the space of \( \mathcal{K} \) solutions, and hence no natural Fock space vacuum. The "correct" choice will be motivated not by mathematics, but by the physical question we are trying to ask in a particular context.

In the case of a black hole, the geometry is stationary, but it is stationary only outside the event horizon. Inside the horizon, the light cones tip over, and the Killing vector becomes spacelike. In this case, the choice of a vacuum for the stationary region outside the horizon amounts to imposing appropriate boundary conditions on the fields outside the horizon.

A situation in which \( \mathcal{K} \) spacetime is not stationary, but there are (two) natural choices for a Fock space vacuum...
is the case of a spacetime that becomes asymptotically stationary in the past or future (or both).

If \( \{u_i, u^x, v_i, v^x\} \) is our standard basis for the solutions to the KG eqn on flat spacetime, then we may choose solutions to the exact KG eqn such that

\[ p_i \rightarrow u_i \text{ in past}. \]

Then \( (p_i, p_j) = \delta_{ij} \) \( (p_i, p_j^x) = 0 \) \( (p_i^x, p_j^x) = -\delta_{ij} \) on any slice, since this is true for a slice in the asymptotic past, and the inner product is independent of the slice. Similarly, we may choose a basis

\[ f_i \rightarrow u_i \text{ in future}. \]

This becomes positive frequency in the future. These two bases need not coincide. A basis of positive freq solns in the past will propagate to a positive norm basis in the future, but not necessarily to a basis of positive frequency solns in the future. Hence, the incoming Fock space, vacuum may evolve to a state that is not vacuum in the future - the fluctuating geometry can create particles.
In this situation, where the space-time is asymptotically stationary (in particular flat) in the past and future on S matrix can be defined that relates the past and future Fock space basis – and hence gives the amplitude for an incoming particle state to scatter off the geometry, yielding an outgoing particle state.

Let |\psi\rangle denote a Fock space state for QFT on flat space-time. Then we denote by

|\psi_{in}\rangle, |\psi_{out}\rangle

The states of the theory on a non-trivial background must asymptotically approach |\psi\rangle in the past, future respectively.

Then we define $S$ by

$$S|\psi_{in}\rangle = |\psi_{out}\rangle$$

$S$ will then be a unitary operator (it preserves the inner product – what comes in goes out).

Thus

$$\langle\psi_{in}| = \langle\psi_{out}|S^*$$

And we have

$$\langle\chi_{in}|\psi_{out}\rangle = \langle\chi_{in}|S|\psi_{in}\rangle$$

This is an inner product between states of the theory on a trivial background, if evaluated on the asymptotic past or future.
Remark: we have defined $S$ as a change of the Fock basis in the Hilbert space $\mathcal{H}$ of the theory.

$1\gamma_{\text{out}} > \in \mathcal{H}$ resembles $1\gamma > \in \mathcal{H}_{\text{out}}$ in future

$1\gamma_{\text{in}} > \in \mathcal{H}$ resembles $1\gamma > \in \mathcal{H}_{\text{in}}$ in past

Then

$S 1\gamma_{\text{out}} > = 1\gamma_{\text{in}} >$

An alternative is to define $S : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{out}}$

so that

$1\gamma_{\text{in}} > = 1\gamma_{\text{out}} > \Rightarrow S 1\gamma > = 1\gamma >$

Then

$1i_{\text{in}} > = S 1i_{\text{out}} > = 1i_{\text{out}} > < j_{\text{out}} | S 1i_{\text{out}} >$

$\Rightarrow S 1i > = 1i > < j_{\text{out}} | S 1i_{\text{out}} >$

and thus

$< j_{\text{out}} | S 1i > = < j_{\text{out}} | S 1i_{\text{out}} >$

$= < j_{\text{out}} | S 1i_{\text{in}} >$

In the language we have used here, we have

$\phi = \sum_{i} (r_{i} a_{i_{\text{in}}}^{\dagger} + r_{i}^{*} a_{i_{\text{out}}}^{\dagger}) = \sum_{i} (r_{i} a_{i_{\text{in}}}^{\dagger} + r_{i}^{*} a_{i_{\text{out}}}^{\dagger})$

(as on page 2.27) where $\phi : \mathcal{H} \rightarrow \mathcal{H}$.

In the alternate language $\tilde{a}_{\text{in}} : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{in}}$

and $\tilde{a}_{\text{out}} : \mathcal{H}_{\text{out}} \rightarrow \mathcal{H}_{\text{out}}$

These are related by

$S \left[ \sum_{i} (r_{i} a_{i_{\text{in}}}^{\dagger} + r_{i}^{*} a_{i_{\text{out}}}^{\dagger}) \right] S^{-1} = \sum_{i} (r_{i} a_{i_{\text{in}}}^{\dagger} + r_{i}^{*} a_{i_{\text{out}}}^{\dagger})$

and so

$S \left( a_{i_{\text{in}}}^{\dagger} a_{i_{\text{out}}} + a_{i_{\text{in}}} a_{i_{\text{out}}}^{\dagger} \right) S^{-1} = \left( a_{i_{\text{in}}}^{\dagger} a_{i_{\text{out}}}^{\dagger} \right) \left( \begin{array}{c} \alpha^{T} \\ \beta^{T} \end{array} \right)$

$\Rightarrow S \left( a_{i_{\text{in}}}^{\dagger} a_{i_{\text{out}}}^{\dagger} \right) S^{-1} = \left( \begin{array}{c} \alpha^{T} \\ \beta^{T} \end{array} \right) \left( \begin{array}{c} \alpha^{\dagger} \\ \beta^{\dagger} \end{array} \right)$
or
\[ S^{-1} \begin{pmatrix} \alpha^* & \beta^* \\ \beta^* & \alpha^* \end{pmatrix} S = \begin{pmatrix} \alpha^* & -\beta^* \\ -\beta^* & \alpha^* \end{pmatrix} \]

This is precisely the same equation that we have for \( S \) on page (2.27), provided we identify \( \alpha^\dagger \) and \( \alpha \). In making the identification, we are recognizing the natural isomorphism \( \mathcal{H}^\dagger \rightarrow \mathcal{H} \), and that we must make use of this isomorphism to define \( S \) matrix elements (since \( \mathcal{H}^\dagger \) and \( \mathcal{H} \) are otherwise regarded as distinct spaces.)
Let \( |i\rangle \) denote a complete basis for the field of theory on trivial BG (not just H, but all of field space).

Then the S-matrix has the representation:

\[
S = \sum_i |i, in\rangle \langle i, in|
\]

(This ensures \( S |i, in\rangle = |i, in\rangle \) acting on basis.)

Therefore,

\[
S^{-1} = S^* = \sum_i |i, out\rangle \langle i, out|
\]

If \( \Theta^{_m} \) is any operator, then

\[
<i, in | \Theta^{_m} |j, in\rangle = <i, out | S^{-1} \Theta^{_m} S |j, out\rangle
\]

or

\[
\Theta^{_m} = S^{-1} \Theta^{_m} S
\]

- i.e. \( S^{-1} \Theta^{_m} S \) has the same matrix element between out states as \( \Theta^{_m} \) does between in states.

We can calculate \( S \) if we can solve the KG equ on the nontrivial BG. The solutions \( f, p \) solve the flat space equ in past and future, so since \( f, p, u, v \) are a complete basis for solutions we have

\[
f_i \rightarrow u_i \text{ in future}
\]

\[
f_i \rightarrow \alpha_{ij} u_j + \beta_{ij} v_j \text{ in past}
\]

for appropriate matrices \( \alpha, \beta \).
Similarly,

\[ p_i \rightarrow u_i \text{ in past} \]
\[ p_i \rightarrow \delta_{ij} u_j + \delta_{ij} u_j^* \text{ in future} \]

But because the KG equation is linear and homogeneous everywhere the \( \Phi \) is nontrivial, we have

\[ f_i = \alpha_{ij} p_j + \beta_{ij} p_j^* \]
\[ p_i = \gamma_{ij} f_j + \delta_{ij} f_j^* \]

By expanding \( \Phi \) in terms of both bases, we find

\[ \Phi = \sum_i (p_i a_i^m + p_i^* a_i^{m+}) \]
\[ = \sum_j (f_j a_j^{m+} + f_j^* a_j^{m+}) \]

and thus infer a linear relation among \( a_i^m \) and \( a_i^{m+} \).

We may then solve for \( S \) by demanding

\[ S^{-1} a_i^m S = a_i^m \]
\[ S^{-1} a_i^{m+} S = a_i^{m+} \]

To see how this goes, let's consider the somewhat more general problem of the relation between two Fock space bases obtained by expanding the field \( \Phi \) in terms of two different orthonormal bases for the KG solns (not necessarily related).
\[ u_i' = a_{ij} u_j + b_{ij} u_j^* \]
\[ u_i'^* = b_{ij}^* u_j + a_{ij}^* u_j^* \]

In matrix notation:
\[
\begin{pmatrix}
u_i' \\
u_i'^*
\end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^*
\end{pmatrix} \begin{pmatrix} u_i \\
u_i^*
\end{pmatrix}
\]

If both bases are normalized so that
\[ (u_i, u_j) = \delta_{ij} \quad (u_i^*, u_j) = 0 \quad (u_i^*, u_j^*) = -\delta_{ij} \]
\[ (u_i', u_j') = \delta_{ij} \quad (u_i'^*, u_j') = 0 \quad (u_i'^*, u_j'^*) = -\delta_{ij} \]

Then we have
\[ \delta_{ij} = (u_i', u_j') = \alpha_{ik}^* \beta_{jk} - \beta_{ik}^* \alpha_{jk} \]
\[ = (\alpha \alpha^* - \beta \beta^*) \delta_{ij} \]

Thus
\[ (\alpha \alpha^* - \beta \beta^*) = I \]

Also
\[ 0 = (u_i'^*, u_j) = \beta_{ik} \alpha_{jk} - \alpha_{ik} \beta_{jk} \]
\[ \Rightarrow \quad \alpha \beta^T - \beta \alpha^T = 0 \]

From these identities, it follows that
\[
(\alpha \beta^* - \beta \alpha^*)^{-1} = \begin{pmatrix} \alpha^* & -\beta^* \\ -\beta & \alpha^* \end{pmatrix} \quad \text{(inverse is unique if it exists)}
\]

And therefore (as for right inverses as well as left inverse) additional identities on p. 2.26)
\[
\begin{pmatrix}
\mathbf{u} \\
\mathbf{u}^* 
\end{pmatrix} =
\begin{pmatrix}
\alpha^+ \\
\beta^+
\end{pmatrix}
\begin{pmatrix}
\beta^T \\
\alpha^T
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}' \\
\mathbf{u}'^*
\end{pmatrix}
\]

Compose now the two mode expansions:
\[
\phi = \sum_i (u_i a_i + u_i^* a_i^+) = (a a^+) (u^*)
\]
\[
= \sum_i (u_i' a_i' + u_i'^* a_i'^+)
\]
\[
= (a' a'^+) (u'^*)
\]

Substituting for \( u'u'^* \) in terms of \( u u^* \) and vice versa gives
\[
(a a^+) (u^*) = (a' a'^+) \begin{pmatrix}
\alpha^x \\
\beta^x
\end{pmatrix}
\begin{pmatrix}
\alpha^* \\
\beta^*
\end{pmatrix}
(u^*)
\]
\[
(a' a'^+) (u'^*) = (a a^+) \begin{pmatrix}
\alpha^+ \\
\beta^+
\end{pmatrix}
\begin{pmatrix}
\beta^T \\
\alpha^T
\end{pmatrix}
(u'^*)
\]

And therefore, since the bases are complete and orthonormal (e.g., we can isolate coefficients by taking KG inner product), we have
\[
(a a^+) = (a' a'^+) \begin{pmatrix}
\alpha^x \\
\beta^x
\end{pmatrix}
\begin{pmatrix}
\alpha^* \\
\beta^*
\end{pmatrix}
\]
\[
(a' a'^+) = (a a^+) \begin{pmatrix}
\alpha^+ \\
\beta^+
\end{pmatrix}
\begin{pmatrix}
\beta^T \\
\alpha^T
\end{pmatrix}
\]

and taking transposes:
\[
\begin{pmatrix}
\alpha' \\
\alpha'^+
\end{pmatrix} = \begin{pmatrix}
\alpha^x \\
-\beta^x
\end{pmatrix}
\begin{pmatrix}
\alpha^+ \\
\beta^+
\end{pmatrix}
\]
\[
\begin{pmatrix}
\beta' \\
\beta'^+
\end{pmatrix} = \begin{pmatrix}
\beta^T \\
\alpha^T
\end{pmatrix}
\begin{pmatrix}
\alpha^+ \\
\beta^+
\end{pmatrix}
\]
Since \((\alpha^T, \beta^+) = (\alpha^*, -\beta^*)^{-1}\), we have the additional identities:

\[
\begin{align*}
\alpha^T \alpha^* - \beta^+ \beta &= 1 \\
\alpha^+ \beta^* - \beta^+ \alpha &= 0
\end{align*}
\]

Since \(\psi, \psi^*\) and \(\psi, \psi^*\) are normalized, both \(\psi, \alpha_+\psi\) and \(\psi, \alpha_+\psi\) satisfy the standard commutation relations. The transformation \((\alpha, \alpha^+) \rightarrow (\gamma, \gamma^+)\) is a canonical transformation - a change of variable that preserves the commutation relations. A canonical transformation that is linear in the \(\alpha_+\psi\) is called a Bogolubov transformation.

For \(\beta \neq 0\), we have changed the positive norm subspace of the space of solutions to \(X_6\). Nonetheless, the two Hilbert spaces are related by a unitary transformation, because a canonical transformation can be implemented by a unitary transformation acting on the Hilbert space.

(Actually, for a system with an \(\infty\) number of degrees of freedom, this is true only subject to certain conditions that will emerge from the calculation below.)

As described on page 2.21, we may define a unitary change of basis by
in which case
\[ U^{-1}aU = a', \]
\[ U^{-1}a^+U = a'^+ \]
(Note: \( U \) will be unitary if it preserves the norm of \( \| \theta \|^2 \). Then, since \( U \) preserves commutators, it preserves normal ordered states.

From these equations and the Bogoliubov transformation, we can solve for \( U \) in terms of \( a \) or \( a' \).

It takes the same form in both representations, since
\[ \langle \theta' | U | \theta' \rangle = \langle \theta' | U | \theta' \rangle. \]

It is convenient to express \( U \) as a normal ordered function of \( a, a^+ \)
\[ U = : U(a, a^+) : \]
(Where double dots denote normal ordering).

Normal-ordered means that all \( a \)'s lie to the right of all \( a^+ \)'s. To solve
\[ aU = Ua' = U(a^*a - \beta^*a^+), \]
\[ a^+U = Ua'^+ = U(-\beta a + \alpha a^+), \]
we note that operators \( a, a^+ \) act on normal-ordered functions as
\[ a_i : \theta' = : (a_i + \frac{2}{\alpha} a_i^+ ) \theta' : \]
\[ a_i^+ : \theta' = : a_i^+ \theta' : \]
\[ : \theta : a_i = : a_i \theta : \]
\[ : \theta : a_i^+ = : (a_i^+ + \frac{2}{\beta} a_i ) \theta' : \]
since  \[ [a_i, f(a^+)] = \frac{2}{\delta a_i} f(a^+) \]

\[ [f(a), a_i^+] = \frac{2}{\delta a_i} f(a) \]

So we must find a function \( \mathcal{U}(\alpha, \alpha^+) \) satisfying

(i)  \( (\alpha + \frac{2}{\delta a^+}) \mathcal{U} = [\alpha x \alpha - \beta \star (\alpha + \frac{2}{\delta a})] \mathcal{U} \)

(ii)  \( \alpha^+ \mathcal{U} = [-\beta \alpha + \lambda (\alpha + \frac{2}{\delta a})] \mathcal{U} \)

(normal ordering now understood and not explicitly indicated).

we solve this by means of the Ansatz

\[ \mathcal{U} = C \exp \left[ \frac{1}{2} (\alpha \alpha^+) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right] \]

\[ = C \exp \left[ \frac{1}{2} (a M_{11} \alpha + a M_{12} \alpha^+ + \alpha^+ M_{21} \alpha + \alpha^+ M_{22} \alpha^+) \right] \]

where we may assume without loss of generality

\[ M_{11} = M_{11}^T, \quad M_{22} = M_{22}^T, \quad M_{12} = M_{21}^T \]

Then

\[ \frac{2}{\delta a} \mathcal{U} = (M_{11} \alpha + M_{12} \alpha^+) \mathcal{U}, \]

\[ \frac{2}{\delta a^+} \mathcal{U} = (M_{21} \alpha + M_{22} \alpha^+) \mathcal{U}, \]
and we have the algebraic equations:

(i) \((a + M_{21} a + M_{22} a^+) = \alpha^* a - \beta^* (a^+ + M_{11} a + M_{12} a^+)\)

(ii) \(a^+ = -\beta a + \alpha (a^+ + M_{11} a + M_{12} a^+)\)

Since the coefficients of \(a\) and \(a^+\) must vanish,

(ii) implies

\[
\begin{align*}
1 &= \alpha + \alpha M_{12} \\
0 &= -\beta + \alpha M_{11}
\end{align*}
\]

\[M_{12} = \alpha^{-1} - I, \quad M_{11} = \alpha^{-1} \beta\]

How do we know that \(\alpha\) is invertible?
This follows from the identity

\[\alpha a^+ = I + \beta \beta^+\]

which shows that \(\alpha a^+\) has no zero eigenvalue. Thus

\[\langle x | \alpha a^+ | x \rangle = (I + \beta \beta^+) |x\rangle^2 > 0;\]

\(a^+\) has trivial kernel, and hence its range is the whole space. Furthermore

\[(a^+ a)^* = I + \beta \beta^+\]

shows that

kernel \(a^+ = 0\) so \(\alpha\) has a left and right inverse.

From (i) we find

\[M_{22} = -\beta^* - \beta^* M_{12} = -\beta^* \alpha^{-1}\]

\[I + M_{21} = \alpha^* - \beta^* M_{11} = \alpha^* - \beta^* \alpha^{-1} \beta\]
\[
\begin{align*}
M_{22} &= -\beta^* \alpha^{-1}, \\
M_{21} &= -I + \alpha^* - \beta^* \alpha^{-1} \beta \\
M_{11} &= M_{11}^T, \quad M_{22} = M_{22}^T, \quad M_{21} = M_{12}^T
\end{align*}
\]

Now we must check that this solution is consistent with the assumptions.

These relations actually follow from the identities...

\[
\begin{align*}
\alpha \alpha^* - \beta \beta^* &= I \\
\beta \alpha^* &= \alpha \beta^T \\
\alpha^T \alpha^* - \beta^+ \beta &= I \\
\beta^+ \alpha &= \alpha^T \beta^* \\
\end{align*}
\]

E.g., from \(\beta \alpha^T = \alpha \beta^T\) we have \(\alpha^{-1} \beta = \beta^T (\alpha \beta^T)^{-1} = (\alpha^{-1} \beta)^T\)

\[
\beta^+ \alpha = \alpha^T \beta^* \Rightarrow \beta^* \alpha^{-1} = (\alpha T)^{-1} \beta^+ = (\beta^* \alpha^{-1})^T
\]

and from \(I = \alpha^T \alpha^* - \beta^+ \beta = \alpha^T \alpha^* - (\alpha^T \beta^* \alpha^{-1}) \beta\),

we have

\[
(\alpha T)^{-1} = \alpha^* - \beta^* \alpha^{-1} \beta
\]

so that \(M_{12} = M_{21}\)

We have now found

\[
U = C : \exp \left[ \frac{1}{2} a(\alpha^{-1} \beta) a + a(\alpha^{-1} \beta) \alpha^T + \frac{1}{2} \alpha^T (\beta^* \alpha^{-1}) \alpha \right]
\]
To complete the calculation of $U$, we must find the constant $C$. We may determine $C$ up to a phase by recalling that $U$ is unitary, so that

$$10' = U^* 10 \Rightarrow \langle 0' 10' \rangle = \langle 010 \rangle = 1$$

where $10$ is the vacuum satisfying $a_i 10 = 0$.

Since the adjoint of a normal ordered operator is normal-ordered, we have

$$U^* = C^* \exp \left[ \frac{i}{2} a^+ (\alpha^{-1} \beta)^* a^+ + \cdots \right]$$

and

$$U^* 10 = C^* \exp \left[ \frac{i}{2} a^+ (\alpha^{-1} \beta)^* a^+ \right] 10$$

Now, we invoke the identity

$$\langle 01 \exp (\frac{i}{2} a M a^+ \exp (\frac{i}{2} a^+ M^* a^+ 10)$$

$$= \left[ \det (I - MM^*) \right]^{-\frac{1}{2}}$$

where $M = M^T$.

(We will return to the derivation of this identity shortly.)

Thus,

$$1 = |C|^2 \left[ \det \left[ I - \alpha^{-1} \beta (\alpha^{-1} \beta)^* \right] \right]^{-\frac{1}{2}}$$

We can simplify the determinant by recalling
The identities
\[ \alpha^* \beta - \beta^T \beta^* = \mathbb{I}, \quad \alpha^* \beta = \beta^T \alpha^* \]
\[ \Rightarrow \mathbb{I} = \alpha^* \alpha - \alpha^* \beta \beta^*(\alpha^*)^{-1} \beta^* \]
\[ \Rightarrow (\alpha^* \alpha)^{-1} = \mathbb{I} - \alpha^{-1} \beta (\alpha^*)^{-1} \beta^* \]

So we have
\[ |C|^2 = (\det(\alpha^* \alpha))^{-\frac{1}{2}} \]

and we have determined \( C \) up to a phase
\[ |C| = (\det(\alpha^* \alpha))^{-\frac{1}{4}} \]

We will not attempt to determine the phase of \( C \). This phase has little physical relevance. Moreover, it is subject to ambiguities concerning how the renormalized energy-momentum tensor \( T_{\mu \nu} \) is defined, as we will discuss later.

In the case of an asymptotically flat spacetime, we have found the S-matrix
\[ S = \text{(phase)}(\det(\alpha^* \alpha))^{-\frac{1}{4}} \exp \left[ \frac{i}{2} a^m (\alpha^{-1} \beta) a^l \right. \]
\[ + a^m (\alpha^{-1} \beta^*) a^l + \frac{i}{2} a^m (\beta^* \alpha^{-1}) a^l \]

It has the same form when expressed in terms of \( a_{\text{out}} \) and \( a_{\text{out}^+} \), since
\[ \langle \psi_{\text{out}} | S | \psi_{\text{in}} \rangle = \langle \psi_{\text{out}^+} | S | \psi_{\text{in}} \rangle \]
We have
\[ |\langle 0\text{out} | 10\text{in} \rangle|^2 = (\det \alpha + \alpha^t)^{-\frac{1}{2}} \]

Note that, in order for the construction of a unitary S-matrix to be possible, we must have \( \det (\alpha + \alpha^t) < \infty \), so \( |\langle 0\text{out} | 10\text{in} \rangle| \) must have a non-zero overlap with \( |10\text{in} \rangle \). We may state this criterion in a somewhat more physical way by noting that
\[ N_i = |\langle 0\text{in} | a_i \text{out} + a_i^\dagger \text{out} | 10\text{in} \rangle| \]
\[ = \langle 0\text{in} | (-B_{ij} a_j^\dagger \chi - B_{ik}^* a_k) | 10\text{in} \rangle \]
\[ = (\beta \beta^+)_{ij} = \sum_j (B_{ij})^2 \]

This is the expectation value of the number of particles of type \( i \) produced by the geometry, when the vacuum state is the vacuum. The total number of particles produced is
\[ N = \sum_i N_i = \tau (\beta \beta^+) \]

And since \( \alpha \alpha^t = \mathbb{I} + \beta \beta^t \), the condition that
\[ \det (\alpha \alpha^t) = \det (\mathbb{I} + \beta \beta^t) < \infty \]
is precisely \( N < \infty \).
The canonical transformation
\[(a^m, a^{m+}) \rightarrow (a^{m+}, a^{m+})\]
cannot be implemented by a unitary operator if the \(10^m\rangle\) vacuum is a state with an indefinite number of particles when expanded in terms of the antifock basis.

(This may happen in principle if there are massless particles, and many soft particles are produced by the fluctuating geometry. It does not happen if the (smooth) spacetime is flat outside a compact region, for fluctuating geometry in a compact region does not produce ordinary soft particles (a theorem P. Wald, Ann. Phys. 118 (1979) 490, "Existence of the S-matrix in QFT in Curved Spacetime").)

Some other S-matrix elements are:
\[\langle i \omega i | j \omega i \rangle = \langle 0 \omega i | 0 \omega i \rangle \left( \frac{1}{1 + \chi^{-1} - 1} \right)_{i,j} = \chi^{-1}_{i,j} \]

An incoming particle may scatter, and emerge with its momentum changed \( \text{if } \chi \neq 1 \)
\( \langle ij \text{ out} | 10 \text{ in} \rangle = \langle 0\text{ out} | 10 \text{ in} \rangle (-\beta X a^{-1})ij \)

\( \langle 0\text{ out} | ij' \text{ in} \rangle = \langle 0\text{ out} | 10 \text{ in} \rangle (a^{-1} \beta)ij' \)

Particles are created or annihilated only if \( \beta \neq 0 \) (Bogolubov transformation mixes pos. and neg. frequencies).

If \( \beta = 0 \) then \( a \) is unitary and \( \langle 0\text{ out} | 10 \text{ in} \rangle \) is a phase.

Particles are not created or annihilated singly, but only in pairs. (These would be particle-antiparticle pairs in the core of a complex scalar field.)

It is also easy to extract from our expression for \( S \) the many-particle \( \rightarrow \) many-particle amplitudes. Consider for example

\( \langle i_1 - i_2 \text{ out} | 10 \text{ in} \rangle \)

\( = \langle i_1 - i_2 | 15 | 10 \rangle = \langle i_1 - i_2 | \frac{1}{n!} (V + V^*)^n | 10 \rangle \)

where \( V = -\beta X a^{-1} \)

allowing the \( 2n \) \( a \)'s to annihilate (to the left) \( K \) \( 2n \) particles \( i_1 - i_2 \) generate \( (2n)! \) terms, each term giving a product

\( V \cdot V \cdot \cdots V \) of \( n \) matrix elements

a \( 2n \times 2n \) symmetric matrix \( V \)

This product depends only on how \( K \) \( 2n \) indices are paired, and each pairing occurs \( n!2^n \) times.
This is the number of the \((2n)!\) permutations of the indices that leave the pairing unchanged. Thus, the \(\frac{1}{n!2^n}\) gets cancelled, and we have

\[
\langle i_1, \ldots, i_{2n} \text{ out} | 0 \text{ in} \rangle
\]

\[
= V_{i_1 i_2} V_{i_3 i_4} - V_{i_2 i_3} V_{i_1 i_4} + \text{all other pairings,}
\]

there being \((2n)! / n!2^n\) terms in the sum.

E.g.,

\[
\langle i_1, \ldots, i_4 \text{ out} | 0 \text{ in} \rangle = V_{i_1 i_2} V_{i_3 i_4} + V_{i_1 i_3} V_{i_2 i_4} + V_{i_1 i_4} V_{i_2 i_3}
\]

We return now to the derivation of the identity

\[
\langle 0 | \exp \left( \frac{1}{2} a^\dagger a \right) \exp \left( \frac{1}{2} a a^\dagger \right) | 10 \rangle
\]

\[
= \left[ \text{det} (1 - M M^\dagger) \right]^{-\frac{1}{2}}
\]

(whence \(M = M^T\)).

The slick way to evaluate such matrix elements is to convert them to Gaussian integrals.

An arbitrary Fock space state can be represented as a function of \(a^\dagger\) acting on \(10\)

\[
| 14 \rangle = \psi (a^\dagger) | 10 \rangle = \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n} \sum_{a_i, \ldots, a_i^\dagger} | a_i^\dagger a_i^{\dagger \ldots a_i^{\dagger \ldots}} a_i^{\dagger} a_i^{\dagger \ldots a_i^{\dagger \ldots}} | 10 \rangle
\]

Claim:

\[
\langle X | 14 \rangle = \frac{\text{Stada}^+ X^\dagger X^a \psi (a^\dagger) e^{-a a^\dagger}}{\text{Stada}^+ e^{-a a^\dagger}}
\]
Here, on the RHS, $a_i a_i^+$ are complex c-nums and
\[ \mathcal{S} \mathcal{d} a_a = \int \mathcal{S} \mathcal{d} a_i d a_i^+ \]
as an integral over
averaged and imaginary part of each $a_i$

we see that to verify the claim, it is enough to show (since we can expand in powers of $a^+$):

\[ \langle 0 | a_i - a_i^+ a_j^+ - a_j^+ | 0 \rangle \]
\[ = \frac{\mathcal{S} \mathcal{d} a_a^+ a_i - a_i^+ a_j^+ a_j^+ e^{-a^+ a}}{\mathcal{S} \mathcal{d} a_a^+ e^{-a^+ a}} \]

we can derive this identity by evaluating both sides.

continuing $a_i$ through $a_j$'s, we find

\[ \langle 0 | a_i - a_i^+ a_j^+ - a_j^+ | 0 \rangle = 0 \quad (m \neq n) \]
\[ = \delta_{ij} - \text{Sinj} + \text{perms} (n \text{! Terms altogether}) \quad (m = n) \]

To evaluate integral on LHS, construct the generating function

\[ Z[J] = \mathcal{S} \mathcal{d} a_a^+ e^{\frac{1}{2} J a e^{a^+} J e^{-a^+ a}} \]

we complete the square.
\[ Z[J] = \text{Sd} \text{d} \text{a} + \exp \left[ -(a^+ - J)(a - J) + J J \right] \]

Shift the integrals:
\[ = e^{J J} Z[0] \]

Now,
\[ \text{Sd} \text{d} \text{a} + a_i a_j - a_i a^+ j - a_j a^+ i e^{-a^+ a} / \text{Sd} \text{d} \text{a} e^{a^+ a} \]

\[ = \frac{1}{Z[0]} \frac{\partial^2}{J J_i} - \frac{\partial^2}{J J_i} J J_j - \frac{\partial^2}{J J_j} Z[J] \bigg|_{J = J = 0} \]

\[ = \frac{\partial^2}{J J_i} - \frac{\partial^2}{J J_j} e^{J J} \bigg|_{J = J = 0} \]

This evidently vanishes for \( n \neq m \). For \( n = m \), we have

\[ = \frac{\partial^2}{J J_i} - \frac{\partial^2}{J J_j} (J_i, \ldots J_i) \]

\[ = J_{i j} J_j - \text{S} i_{j j} J_j + \text{perms} \]

---

By means of this trick, we have

\[ \langle 0 \mid \text{exp} \left[ \frac{i}{2} a^+ M a \right] \text{exp} \left[ \frac{i}{2} a^+ M^* a \right] \mid 0 \rangle \]

\[ = \text{Sd} \text{d} \text{a} + \exp \left[ \frac{i}{2} (a a^+) \left( \begin{array}{cc} M & \Pi \\ \Pi^* & M^* \end{array} \right) (a a^+) \right] \text{Sd} \text{d} \text{a} + \exp (a^+ a) \]
You can evaluate such an integral by changing variables to real
\[ \begin{align*}
X &= \frac{1}{\sqrt{\pi}} (a + at) \\
Y &= \frac{i}{\sqrt{\pi}} (a - at)
\end{align*} \]

A real Gaussian integral
\[ S dX e^{-\frac{1}{2} X \cdot A X} \quad (A = AT) \]

may be evaluated if \( A \) has an orthonormal basis of eigenvectors
\[ A e_n = \lambda_n e_n \]

By expanding \( X = \sum_n X e_n \) in this basis, we find
\[ \frac{S dX e^{-\frac{1}{2} X \cdot A X}}{S dX e^{-\frac{1}{2} X \cdot X}} = \prod_n \frac{S dX_n e^{-\frac{1}{2} X_n \lambda_n X_n}}{S dX_n e^{-\frac{1}{2} X_n^2}} \]
\[ = \prod_n \lambda_n^{-\frac{1}{2}} = (\text{det} A)^{-\frac{1}{2}} \]

The integrals converge for \( \text{Re} \lambda_n > 0 \), and may be defined by analytic continuation otherwise.

Writing our integral in terms of real variables and using elementary properties of determinants, we find (exercise)
\[ \langle 0 | \exp \left( \frac{i}{2} a M a^+ \right) \exp \left( \frac{i}{2} a^+ M^* a \right) | 0 \rangle = \left[ \text{det} \left( M^{-\frac{1}{2}} \right) \right]^{-\frac{1}{2}} \]
\[ \det \begin{pmatrix} \mathbb{1} & M^* \\ M & \mathbb{1} \end{pmatrix} = \det \begin{pmatrix} \mathbb{1} & 0 \\ -M & \mathbb{1} \end{pmatrix} \det \begin{pmatrix} \mathbb{1} & M^* \\ M & \mathbb{1} \end{pmatrix} \]

\[ = \det \begin{pmatrix} \mathbb{1} & M^* \\ 0 & \mathbb{1} - MM^* \end{pmatrix} = \det (\mathbb{1} - MM^*) \]

—which was to be shown.