Yang-Mills in (3+1) dimensions

To what extent can our insight into confinement in (2+1) dimensions be extended to (3+1) dimensions?

In (3+1) dimensions, there is no conserved vortex number in the Higgs phase, and so strictly speaking no Z_N topological symmetry (at least at zero temperature). Still, some of the preceding discussion can be generalized.

Although the vortex is not a stable particle, there is a locally stable magnetic flux tube in the Higgs phase, and an operator \( B(C) \) - the "Closed Loop" - that can create or destroy strings of flux.

We can construct the "Closed Loop" operator \( B(C) \) the same way we construct the vortex operator \( B(x) \). In the canonical language, acting on a time slice in \( A^0 = 0 \) gauge is a gauge transformation, trivial at spatial infinity, singular on \( C \), and with winding by \( e^{2\pi i/N} \in \mathbb{Z}_N \) on a closed path that links once with \( C \) in the sense defined by the right-hand rule.
Therefore, \( B(1C) \) creates an infinitesimally \( \mathbb{Z}_N \) magnetic (anti) flux. As before, the action of \( B(1C) \) on physical states depends only on the nature of the singularity at \( C \) of the gauge transformation \( \Phi(1C) \):

\[
B(1C) |\overline{A}_i(x)\rangle = |\overline{A}_i(x)\rangle^{\Phi(1C)}
\]

We can see that

\[
[B(1C), B(1C')] = 0
\]

if the loops \( C \) and \( C' \) don't intersect, and that

\[
[B(1C), \Omega(R)] = 0 \quad \text{if} \quad C \cap \Omega = \emptyset
\]

if \( \Omega \) is a gauge invariant region smeared in a region \( R \) that does not intersect \( C \).

In the Euclidean path integral description, any monopole \( B(1C) \) in the path integral restricts the sum over histories to those constrained so that \( K_a \) is a world line of a singular \( \mathbb{Z}_N \) monopole on the closed loop \( C \).

Hence \( B(1C) \) in spacelike \( C \) creates a closed line of electric \( \mathbb{Z}_N \) flux, and for timelike \( C \) represents an insertion of a classical source carrying \( \mathbb{Z}_N \) electric charge on worldline \( C \).
And $B/C$ for spacelike $C$ creates a closed line of $\mathbb{Z}_N$ magnetic flux, while for timelike $C$ it represents an insertion of a classical $\mathbb{Z}_N$ monopole source.

Thus the Wilson loop $W(C)$ and 't Hooft loop $B(C)$ are really on the same footing, and are related by an electromagnetic duality transformation that interchanges electric and magnetic fields.

As in $2+1$ dimensions, it is instructive to consider the correlation function

$$\langle W(C) B(C) \rangle,$$

which as before is multivalued.

There is an Aharonov-Bohm interaction between a $\mathbb{Z}_N$ charge and a $\mathbb{Z}_N$ magnetic flux tube, or between a $\mathbb{Z}_N$ magnetic monopole and a $\mathbb{Z}_N$ electric flux tube.

Imagine deforming $C$, so that it sweeps out a closed world sheet $\Sigma$.

In 4-dim Euclidean space, the surface $\Sigma$ has a linking number with the loop $C$. 

ς
This is the (signed) number of times that the string \( C' \) winds around the curve on the worldline \( C \). The deformation \( C' \), start and end with the loop \( C' \), then has the effect on the correlation function:

\[
2\pi i k (N) \langle W(0) B(1') \rangle \rightarrow e^{i2\pi k} \langle W(0) B(1') \rangle
\]

where \( k \) is the linking number of \( C' \) with \( C \). Thus the correlation function is multivalued. Alternatively and equivalently, we can imagine carrying \( C' \) around a closed surface \( \Sigma \) that links with \( C' \) — then an electric string winds around a magnetic monopole, and again \( k \) is a non-scalar phase that depends on the linking number.

Now imagine that \( C' \) and \( C \) are at all times for them one another, and consider how \( \langle W(0) B(1') \rangle \) behaves as \( C' \) winds around \( \Sigma \). The fluctuations near \( C \) and \( C' \) should look like fluctuations in the vacuum, so we expect the correlator to factorize

\[
\langle W(0) B(1') \rangle \sim e^{i\theta} \langle W(0) \rangle \langle B(1') \rangle
\]

— up to a phase that may vary as \( C' \) is distorted.
Here $\theta \in [0, 2\pi)$ parameterizes a one-parameter family of loops, beginning and ending with $C$. We know that $\alpha$ must increase by $\frac{2\pi}{N}$,

$$\alpha(\theta + 2\pi) = \alpha(\theta) + \frac{2\pi}{N} \quad \text{mod} \ 2\pi$$

If $C'$ links once with $C$.

If the K"{a}ring has nonlinear excitations, $\theta$ can certainly increase smoothly, but this is not possible if the K"{a}ring has a nonzero gap, and $C, C'$ are for apart compared to the correlation length $\xi$.

What must happen instead is that $\alpha(\theta)$ jumps suddenly. A sudden jump occurs if there is a physically observable sheet stretched across either $C$ or $C'$.

In the former case, the jump occurs as $C'$ crosses the sheet, in the latter case, when $C$ crosses the sheet.

If there is a sheet stretched across $C$, it has an area of the sheet, and so we find
\[ \langle \Phi(x) \rangle = \exp \left[ - \frac{1}{g^2} \langle A(x) \rangle \right] \]

- electric confinement

If there is a sheet stretched across \( C \), then

\[ \langle B_1(x) \rangle = \exp \left[ - \mu \langle \text{Area} \rangle \right] \]

- magnetic confinement

We conclude, in purely topological reasons, that if the theory does not have stable magnetic flux tubes that confine quark modes (is not magnetically ordered), then it must have stable electric flux tubes that confine quarks (as confining).

In 2+1 dimensions, we can distinguish Higgs and confinement phases with the local gauge invariant index found:

\[ \langle \Phi(x) \rangle = 0 \Rightarrow \text{Higgs} \]
\[ \langle \Phi(x) \rangle \neq 0 \Rightarrow \text{Confinement} \]

Alternatively, we could consider

\[ \lim_{|x-y| \to \infty} \langle \Phi(x) \Phi(y) \rangle \]

Although we don't have a local

order parameter in 3+1 dimensions, the theta term provides a similar criterion:
\[ \lim_{\langle B/C \rangle} \sim \begin{cases} \exp(-\mu \cdot \text{Area}) & \Rightarrow \text{Higgs} \\ \exp(-\mu \cdot \text{Perimeter}) & \Rightarrow \text{confinement} \end{cases} \]

- In fact, though, we have not yet excluded the possibility of Higgs and confinement - i.e. a phase with simultaneous area-law behavior in both \( B/C \) and \( A/C \).

**Electric-Magnetic Duality**

We can go further by exploiting more fully the duality symmetry relating \( W(1) \) and \( B(1) \).

We can probe for electric and magnetic confinement by studying the energetics of gauge fields on a 3-torus - i.e. a periodically identified box.

There are topological electric and magnetic fluxes associated with each fundamental non-trivial cycle of the 3-torus. If the ground-state energy in the sector with, say, electric flux running around the 3rd edge of the box in the \( E_3 \) direction grows linearly with the size of the box, then these are stable electric fluxes.
flux tubes. Similarly for magnetic flux:

\[ E \propto \frac{1}{a^3} \sim \frac{\Phi^2}{a} \to 0 \quad \text{as} \quad a \to \infty \]

If flux is confined

\[ E \propto k a \to \infty \quad \text{as} \quad a \to \infty. \]

We have discussed magnetic flux on a 2-torus before. For \( G = SU(1)/\mathbb{Z}_N \), consider a fundamental Wilson loop \( C \) wrapped around the torus. As \( C \) winds, sweeping out the torus and returning to original loop, the phase of \( W(C) \) can change by \( 2\pi i k/N \) for \( k \) \((\mod N)\) as \( k \) \(\mod N\) magnetic flux on the torus.

It is convenient to describe this in a different way...
On the periodically identified box, gauge invariance quantities should be smooth, but the connection $A$ need not be single valued: it can be periodic up to a gauge transformation.

\[ A(a_1, x_2) = A(0, x_2) \]
\[ A(x_1, a_2) = A(x_1, 0) \]

Then we can express $A(a_1, a_2)$ in terms of $A(0, 0)$ in two different ways:

\[ A(a_1, a_2) = \Omega_2(a_1) \Omega_1(a_2) = \Omega_1(0) \Omega_2(a_1) \]
\[ A(a_1, a_2) = \Omega_1(a_1) \Omega_2(0) \Omega_1(a_2) \]

Consistency requires that $\Omega_1(0) \Omega_2(a_1)$ and $\Omega_1(a_1) \Omega_2(0)$ are equivalent as $\text{SU}(N)/\mathbb{Z}_N$ transforming -- i.e., differ by an element of the center of $\text{SU}(N)$ (couple condition).

\[ \Omega_1(0) \Omega_2(a_1) = \exp \left[ -2\pi i n_1/N \right] \Omega_2(0) \Omega_1(a_1) \]
\[ \Omega_1(0) \Omega_1(a_1) \Omega_2(a_2) \Omega_2(0)^{-1} = \exp \left[ -2\pi i n_2/N \right] \]
This element of the center is a topological invariant of the bundle and in fact it is just the magnetic flux.

E.g. suppose an \( U(1) \) vector current operator acts on the torus.

In a particular gauge, the \( U(1) \) gauge transform \( \mathbf{R}(x) \) has no effect on \( \mathbf{R}(x) \), but on \( \mathbf{R}(0) \), but it induces

\[
\mathbf{R}(0) \to e^{-2\pi i / N} \mathbf{R}(0)
\]

and so reduces \( n \rightarrow n - 1 \) — i.e., annihilates a unit of flux.

On the three-torus, there is a flux \( \mathbf{H} \) associated to each of the 3 planes (the homotopically nontrivial surfaces). We take

\[
\mathbf{H} = \Sigma_{ijk} H_{ijk}
\]

to be the magnetic flux in the \( \mathbf{E}_i \) direction.

Electric Flux

What is electric flux? As discussed previously, on a torus there can be "gauge" time-independent gauge transformations that cannot be smoothly deformed to trivial, and so are not required by the Gauss law...
constant to act trivially on physical states. These are the gauge transformations that have a $\mathbb{Z}_N$ winding number—i.e., are periodic up to an element of the center of the gauge group.

On the 3-torus, the gauge transformation $\Sigma$ has three winding numbers:

$$\Sigma(a_1, x_2, x_3) = e^{2\pi i a_1 / N} \Sigma(0, x_2, x_3)$$
$$\Sigma(x_1, a_2, x_3) = e^{2\pi i a_2 / N} \Sigma(x_1, 0, x_3)$$
$$\Sigma(x_1, x_2, a_3) = e^{2\pi i a_3 / N} \Sigma(x_1, x_2, 0)$$

How a gauge transformation acts on physical states is determined only by the values of $a_1, a_2, a_3 \mod N$—as two transformations with the same values differ by a small transformation that acts trivially on a physical state.

Thus we identify a $(\mathbb{Z}_N)^3$ global symmetry for $G=SU(N)/\mathbb{Z}_N$ gauge fields in the box, and states can be decomposed as irreducible representations of the symmetry, in which $\Sigma(K)$ is represented by 

$$\Sigma(K) |\vec{e}\rangle = \exp\left(\frac{2\pi i}{N} \vec{e} \cdot \vec{K}\right) |\vec{e}\rangle$$

where $e_1, e_2, e_3$ are integers defined mod $N$. These integers are the electric fluxes in the $1, 2, 3$ directions.
We can see that $\mathbb{Z}$ is the electric flux by noting that gauge transformations do not commute with the fundamental Wilson loop $W(C)$:

$$W(C) \mathbb{Z}(k^2) = e^{-2\pi i k^3 R(k^2)} W(C)$$

if $C$ winds around toms in the 3 direction (e.g. changes $a_3 \to a_3 + 1$).

So $W(C)$ boosts the eigenvalue of $\mathbb{Z}(k^2)$ by $e_3 \to e_3 + 1$.

This is right, since we know that $W(C)$ creates a line of T files along $C$.

We can construct projection operators $K^{\pm}$ project on states with definite transformation properties, and hence definite flux

$$P(k^2) = \frac{1}{N^3} \sum_{k^2} e^{-2\pi ik^2 \mathbb{Z}(k^2)}$$

-Note $K^{+}$:

$$\mathbb{Z}(k^2) P(k^2) = \frac{1}{N^3} \sum_{k^2} e^{-2\pi ik^2 \mathbb{Z}(k^2)} \mathbb{Z}(k^2 + k^2)$$

by shifting Kosm. This shows $K^{+}$
$(\mathbf{E})$ acting on any state gives a state with electric flux $\mathbf{E}$. It is a projector because

$$P(\mathbf{E}^+) P(\mathbf{E}^-) = \frac{1}{n^3} \frac{\pi^2}{\xi^2} \int d^3 \mathbf{k} \int d^3 \mathbf{\tilde{k}} \frac{\int d^3 \mathbf{\tilde{E}}}{\mathbf{E}} P(\mathbf{E})$$

**Energy of a Flux Configuration**

We will evaluate the ground state energy in a sector of definite flux by inserting into the path integral a projector onto a sector of definite flux, and studying the behavior as the box gets large.

Actually, it is convenient to introduce periodic boundary conditions in the Euclidean time direction and so evaluate the partition function for gauge fields in a box at finite (but slow) temperature.

At inverse temperature $\beta$,

$$Z(\beta) = \frac{\mathrm{Tr} e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}} = \sum_n \langle n | e^{-\beta H} | n \rangle$$

where $\{|n\rangle\}$ is a complete set of field eigenstates.

$$= \sum_{BC} e^{-S_{\beta C}}$$

where the BC imposes periodicity in time direction with period $\beta$. 
In a gauge theory, we need to restrict to the gauge-invariant states, which we can do by inserting a projector, or equivalently, integrating over all small gauge transformations

\[ Z(\beta) = \int dA_i(x) d\bar{S}_i(x) \left< A^S \left| e^{-\beta H} \right| A \right> \]

integrated over small gauge transformations

\[ Z(\beta) = \int (dA_i) e^{-SE[A]} \]

integrated over histories periodic in Euclidean time up to a small gauge transformation.

This \( Z(\beta) \) includes contributions from all flux sectors. We restrict to a particular sector by projecting onto that sector

\[ Z(e^2, m; \beta) = \int dA_i(x) dN_i(x) \left< A^S \left| e^{-\beta H} P(e^2, m) \right| A \right> \]

We project out \( m \) by restricting \( A_i(x) \) to obey the required boundary condition, and project onto \( e^2 \) with

\[ P(e^2) = \frac{1}{N^2} \sum_2 e^{-2\pi i (e^2, \bar{r})/N} \Re \bar{r} \]
Note the $R(\vec{R})$ action on the initial configuration is not the final one – it has the effect of introducing a unit of flux in the $x$-$t$ plane. Thus $\Phi_{oi} = K_i$ is the flux in the $oi$ plane.

And so we have

$$Z[\vec{e}, \vec{m}; \beta] = \frac{1}{N^3} \sum \frac{e^{-2\pi i/N(\vec{e} \cdot \vec{m})}}{\prod_{j=1}^{N} (\vec{e} \cdot \vec{R}_j)} e^{-S_{\text{EA}}[\vec{e}, \vec{m}]} = \frac{1}{N^3} \sum \frac{e^{-2\pi i/N(\vec{e} \cdot \vec{m})}}{\prod_{j=1}^{N} (\vec{e} \cdot \vec{R}_j)} Z[\vec{e}, \vec{m}; \beta]$$

put integrals over sector with $\Phi_{oi} = K_i$

$$MK = \frac{1}{2} E_{Kij} \bar{N}_{ij}$$

We will eventually take the limit

$$\lim_{\beta \to 0} Z[\vec{e}, \vec{m}; \beta] = \lim_{\beta \to 0} e^{-\beta E_{\text{vac}}[\vec{e}, \vec{m}; \vec{a}]} \exp(-\beta E_{\text{vac}}[\vec{e}, \vec{m}; \vec{a}])$$

-- dominated by vacuum in the sector, if there is a mass gap.
Now we can use an electric-magnetic duality relation to constrain $\mathcal{Z} [\vec{E}, \vec{m}]$. We may as well take all sides of the box to be equal: $a_1 = a_2 = a_3 = \beta$. Then $\mathcal{W} [K, m]$ is invariant under Euclidean rotations of the box — if we transform $K, m$ suitably.

E.g. under $K_2$ 4D rotation under which

\[
\begin{bmatrix}
  x_1 & x_2 \\
  x_2 & x_1 \\
  x_3 & x_0 \\
  x_0 & x_3
\end{bmatrix}
\]

\[
\begin{align*}
  x_1 &\rightarrow x_2 \\
  x_2 &\rightarrow x_1 \\
  x_3 &\rightarrow x_0 \\
  x_0 &\rightarrow x_3
\end{align*}
\]

\[
\begin{align*}
  N_{01} &\rightarrow N_{32} = -N_{32} \\
  N_{02} &\rightarrow -N_{21} \\
  N_{03} &\rightarrow -N_{30} = N_{03} \\
  N_{12} &\rightarrow -N_{21} = N_{12} \\
  N_{23} &\rightarrow -N_{10} \\
  N_{31} &\rightarrow -N_{02} = N_{20}
\end{align*}
\]

Thus the rotation leaves $K_3$ and $m_3$ alone, but

\[
K_1 \leftarrow m_1, \\
K_2 \leftarrow m_2
\]

Hence,

\[
\mathcal{W} [K, m] : = \mathcal{W} [m_1, m_2, K_3; K_1, K_2, m_3]
\]

What does this imply for $\mathcal{Z} [\vec{E}, \vec{m}]$?

\[
\mathcal{Z} [\vec{E}, \vec{m}] = \frac{1}{N^3} \sum_{\vec{K}} \mathcal{W} [\vec{E}, \vec{m}, \vec{K}] = \frac{1}{N^3} \sum_{\vec{K}} \mathcal{W} [m_1, m_2, K_3; K_1, K_2, m_3]
\]

Use the inverse transform

\[
\mathcal{W} [K, m] : = \sum_{\vec{E}} \mathcal{Z} [\vec{E}, \vec{m}]
\]
\[ Z[E, \overrightarrow{m}] = \frac{1}{N^2} \sum_{k_1, k_2} \exp \left[ \frac{-2\pi i}{N} (k_1 + k_2) \cdot \overrightarrow{m} \right] \]

The \( k_3 \) sum imposes \( l_3 = e_3 \Rightarrow \)

\[ Z[E, \overrightarrow{m}] = \frac{1}{N^2} \sum_{k_1, k_2} \exp \left[ \frac{-2\pi i}{N} (k_1 e_1 + k_2 e_2 - l_1 m_1 - l_2 m_2) \right] \cdot Z[l_1, l_2, e_3; k_1, k_2, m_3] \]

- The duality relation

What are the consequences? As the box gets large

\[ Z[E, \overrightarrow{m}] \rightarrow \begin{cases} 1 & \text{light flux} \\ 0 & \text{heavy flux} \end{cases} \]

- i.e. \( F \rightarrow 0 \) and \( Z \rightarrow 1 \) if there are no stable flux tubes in that sector, and \( F \rightarrow \infty \) and \( Z \rightarrow 0 \) if there are stable flux tubes.

So \( Z[E, \overrightarrow{m}] \) are \( N \) numbers taking values \( 0, 1 \).

What do we know about these numbers?
We assume \((\vec{a}, \vec{b})\) is a light flux (in vacuum).

If \((\vec{c}, \vec{m})\) is light, then \((0, \vec{m})\) is light.

This is because

\[
Z[\vec{c}, \vec{m}] = \frac{1}{N^3} \sum_{\vec{r}} e^{(2\pi i/\hbar) \vec{r} \cdot \vec{c}} W[\vec{r}, \vec{m}]
\]

Each \(W[\vec{r}, \vec{m}]\) is nonnegative — if the vacuum angle \(\Theta = 0\) — since for the Euclidean path integral \(\Theta\) is positive definite. If \(Z[\vec{c}, \vec{m}] = 1\), then since continuity of \(W\) implies

\[
Z[0, \vec{m}] \geq \frac{1}{N^3} \sum_{\vec{r}} W[\vec{r}, \vec{m}] \Rightarrow Z[\vec{c}, \vec{m}]
\]

we must have \(Z[0, \vec{m}] = 1\), too.

Let's also assume spatial rotational invariance:

\[
Z[\vec{c}, \vec{m}] = Z[\vec{c}, \vec{m}']
\]

If \(R\) is a rotation that preserves the 3D box. Thus

\((\vec{c}, \vec{m})\) in light \(\Rightarrow (R\vec{c}, R\vec{m})\) is light.
From duality, we have the inequality
\[ \mathbb{E}[\vec{e}, \vec{m}] = \frac{1}{N^2} \sum_{\vec{k}\in \mathbb{Z}} \exp \left( -\frac{2\pi i}{N} k_1 e_1 + k_2 e_2 - \ell_1 m_1 + \ell_2 m_2 \right) \]
\[ \leq \frac{1}{N^2} \sum_{\vec{k}\in \mathbb{Z}} \mathbb{E}[\vec{e}, \ell, \ell, 0, 0, 0; \vec{k}_1, \vec{k}_2, m_3] = \mathbb{E}[0, 0, e_3; 0, 0, m_3] \]

Hence \((\vec{e}, \vec{m})\) is light \(\Rightarrow (0, 0, e_3; 0, 0, m_3)\)

Now suppose \((\vec{e}, \vec{m})\) is light, consider the duality relation for \(\mathbb{E}[0, 0, e_3; 0, 0, m_3]\)

\[ J = \mathbb{E}[0, 0, e_3; 0, 0, m_3] = \frac{1}{N^2} \sum_{\vec{k}\in \mathbb{Z}} \mathbb{E}[\ell, \ell, e_3; \ell, \ell, m_3] \]

Since the sum is 1, there must be \(N^2\) terms in the sum with \(J = 1\)

If \((\vec{e}, \vec{m})\) is light, then there are \(N^2\) light fluxes with the same \(e_3\) and \(m_3\)

Now look back at the duality relation for \(\mathbb{E}[\vec{e}, \vec{m}]\) again. There are exactly \(N^2\) terms not all zero, hence each term must be 1, and we conclude that all fluxes are equal.

Hence: if \((e_1, e_2, e_3; m_1, m_2, m_3)\) and \((e'_1, e'_2, e'_3; m'_1, m'_2, m'_3)\) are both light, then

\[ e_1 m_2 + e_2 m_1 - e'_1 m_1 - e'_2 m_2 \equiv 0 \pmod{N} \]
Now suppose \( (\vec{e}, \vec{m}) \) is light.

From above, both
\[
(00, e_3, 00, m_3)
\]
and
\[
(00, 00, 00, m_3)
\]
both have rotational invariance, each
\[
(e_i; 0, 0, m_i, 0, 0)
\]
where \( i, j = 1, 2, 3 \)
and
\[
(0, 0, 0; m_j, 00)
\]
i.e. \( e_i, m_j \) are any component of the light flux.

Hence
\[
\{ e_i, m_j \equiv 0 \pmod{N}, \ i, j = 1, 2, 3 \}
\]

Similarly, if \( (\vec{e'}, \vec{m'}) \) and \( (\vec{e''}, \vec{m''}) \) are light, then so are
\[
(e_i; 00, m_i, 00) \Rightarrow \{ e_i, m_j' \equiv 0 \pmod{N} \}
\]
and so are
\[
(e_i, 00; m_i', 00) \Rightarrow \{ e_i, m_j \equiv 0 \pmod{N} \}
\]

Now suppose \( N \) is prime (e.g. 3).

Then, if \( m \) has a non-zero component in any light flux, \( e = 0 \), and if \( e \) has a non-zero component, \( m = 0 \).
Now, $N'$ prime, there are only 2 rotationally invariant solutions

(i) $(0, m)$ ni light

This is electric confinement

(ii) $(e, 0)$ is light

This is magnetic confinement, i.e., Higgs phase

If $N$ not prime, there can be other solutions (HW exercise).

Naturally, we cannot rule out the Higgs phase, which really is possible with adjoint Higgs fields. And we needed the assumption of a mass gap ($Z = 0$ or 1) to rule out a Coulomb phase. E.g., $SU(2)/Z_2 → U(1)$ has neither electric nor magnetic confinement.
Deconfinement at Finite Temperature

Since we now know how to formulate

$$ Z[\beta, \vec{a}] = \exp \left[-\beta E(\beta, \vec{a})\right] $$

in a gauge theory, we can study the $\vec{a} \to \infty$ limit with $\beta$ fixed, the bulk thermodynamics of the SU(N)/U(1) gauge theory.

Suppose the theory is in the confining phase at $\beta = \infty$. What do we expect to happen as we heat the system up, i.e. increase $\beta$?

Closed loops will arise due to thermal fluctuation.

At high enough temperature, a "confinement" of strings might form.

For adding quark sources might not appreciably increase the free energy – adding another flux tube is not costly if many long flux tubes are already present.
A crude model: string bits occupy a cubic lattice with lattice spacing \( a \) (an flux tube thickness)

The number of paths on the lattice of length \( L \), in \( d \) dimensions

\[
\sim (2d-1)^{L/a}
\]

Paths can turn in \( 2d-2 \) directions

Non self intersection corrects this to

\[
\left( \mu d \right)^{L/a}
\]

\[\mu_2 \sim 2.638
\]

\[\mu_3 \sim 4.684
\]

\[\mu_4 \sim 6.77
\]

The partition function for a gas of string loops behaves like

\[
Z \sim \sum_L \left( \mu d \right)^{L/a} e^{-\beta (KL)}
\]

where \( K \) = string tension. The sum diverges (long strings become unsuppressed)

for

\[
\beta (KL) = \frac{1}{a} \ln \mu d
\]

or

\[
\beta c^{-1} = \frac{a K}{\ln \mu d}
\]

In reality, at \( \Delta^{-1} \), \( K / \Delta^2 \Rightarrow \beta c^{-1} \sim \Delta^{-1} \)
What is a proper criterion for confinement? Finite $\beta$ breaks the symmetry between space-like and time-like loops, and there is no reason to expect it to behave in the same way...

Consider limit of high temperature in $A_0 = 0$ gauge formulation

$$Z[\beta] = \int dA(x) \exp \left[ -\frac{1}{2g^2} \int_0^\beta d\tau \int d^4x \left( A_i^2(x\tau) + (\partial_\tau A_i(x\tau))^2 \right) \right]$$

B.C.: $A_i(x, 0) = 0$

In limit $\beta \to 0$, it is costly for fields to vary with time -- costs kinetic energy $\frac{(\Delta A_i)^2}{\beta}$

As $\beta \to 0$ (high temperature), path integral is dominated by time-independent configurations and the integral over $A(x)$ becomes peaked at trivial SU(N)/ZN gauge transformations -- $Z[\beta]$ is nearly constant and close to an element of $ZN$ -- the center of $SU(N)$.

Expectation values in the thermal ensemble approach those of 3d U(1) Euclidean Yang-Mills theory, with 3-dimensional coupling

$$g_{3d}^2 = g_{4d}^2 \frac{\tau_1}{4} \quad (\tau_1 = \beta^{-2} = \text{temp.})$$
We expect that pure Yang-Mills in 2+1 dim is confining, so sporelike loops Wilson loops in high-temp 3+1 dim YM expected to exhibit area law. Flux tubes are locally stable—but does this imply confinement?

We should consider, instead the response to introducing colored sources via the Dornold path. It is convenient to express the partition function differently—in a gauge with $A_0 \neq 0$ and $A_{\parallel}$ periodic (rather than periodic up to a gauge transformation, in Euclidean time with
period $\beta$):

$$Z[\beta] = \left( \det \Delta_{\parallel} \right) \text{periodic} \quad e^{-\beta S_E}$$

(where gauge fixing understood, to remove infinite volume of gauge group). This becomes earlier's expression if we fix $A_0 = 0$ gauge.

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Actually we can't quite fix $A_0 = 0$ everywhere, since

$$W(0) = \text{Pexp}(i \int A \cdot d\tau)$$

is gauge-invariant—under any finite winding around spacetime in Euclidean time direction.
The best we can do is squeeze a region where $A_0 \neq 0$ into a thin slab. Alternatively, we can set $A_0 = 0$ with a gauge transformation $A_0 \neq 0$ is not periodic.

$$\text{Pexp}(i \int A_0 dt) = \left[ \text{SL}(x^2, t) \right] \left[ \text{Pexp}(i A) \right] \text{SL}(x^2, 0)$$

$$= I$$

$$\Rightarrow \text{SL}(x^2, t) \text{SL}(x^2, 0) = \text{Pexp}(i \int A^2)$$

Then in this $A_0 = 0$ gauge, $A^2(x^2, t)$ is not periodic, but instead periodic up to an arbitrary gauge transformation — and the $A_0$ integral becomes $\int d^2 x^2$.

Now response to introducing sources at $\mathcal{S}$ is probed by

$$\langle \mathcal{W}(C) \mathcal{W}(C') \rangle \beta$$

where $C, C'$ move in time direction. Even more simply

$$\langle \mathcal{W}(C) \rangle = \exp(-\beta \text{F_{quark}})$$
the timelike Wilson line determines the free energy cost of introducing a single classical quark source.

If quarks are confined, then

\[ \text{Quark} (\beta) \rightarrow \infty \text{ in limit of infinite volume,} \]

and

\[ \langle W(x) \rangle_{\beta} \xrightarrow{\text{vol} \rightarrow \infty} 0 \]

Hence, \[ \langle W(x) \rangle_{\beta} \neq 0 \] signifies that there is no "electric disorder" so we can introduce a quark with much cost.

(Adding another long flux tube doesn't matter, because there are a lot of long flux tubes present already.)

As in (2+1) dimensional Yang-Mills at zero temperature, the confinement/deconfinement transition can be interpreted as a change in the realization of a ZN symmetry.

Now the ZN consists of gauge transformations in Euclidean spacetime that are large — i.e., twist by an element of the center in the
Euclidean time direction. These gauge transformations preserve the action and path integral measure, but transform

\[ W(x) \rightarrow e^{2\pi i K/N} W(x) \]

In the confining phase, \( \langle W(x) \rangle = 0 \), and the symmetry is manifest. In the deconfined phase \( \langle W(x) \rangle \neq 0 \), and the \( \mathbb{Z}_N \) is spontaneously broken.

Furthermore, we have already argued that the integral over \( \mathcal{S}_N(x) \) (the remnant of \( W(x) \)) in the \( A_0 = 0 \) gauge) is sharply peaked around

\[ \mathcal{S} \in \mathbb{Z}_N \approx \{ e^{2\pi i k/N}, k = 0, \ldots, N-1 \} \]

in the limit of temperature \( T \rightarrow 0 \).

This is the symptom of spontaneous breaking of the \( \mathbb{Z}_N \) symmetry. Electrolytes in which \( W(x) \) jumps from one near element of the center to near another element become suppressed.

\( \mathbb{Z}_N \) is spontaneously broken \( \langle W(x) \rangle \neq 0 \), and deconfinement occurs. However, spore-like loops still decay like \( e^{-\beta v} \), and there is no magnetic confinement either.
Order of the Transition

Thus the deconfinement transition is in the same universality class as that of a $Z_2$ Ising spin system in 3 dimensions (for $G=SU(2)/Z_2$).

The phase transition should be second order, with Ising exponents. This is confirmed by lattice Monte Carlo studies.

For $G=SU(3)/Z_3$, it is in the universality class of a $Z_3$ magnet, i.e., 3-state Potts model. There is no known fixed point of the renormalization group in this case, and so the transition is expected to be first order. This is also seen in lattice calculations.

There is a heuristic interpretation of the difference between $N=2$ and $N=3$. For $N=3$, flux tubes are directed, and can meet at vertices.

At high temp, instead of a gas of loops, the flux tubes form a percolating network that fills space.
Then if we add one quark, it is pretty easy for the extra flux tube to "hook on" to the existing network, and the network carries the new unit of flux off to $\infty$, without much cost in free energy.

The transition is first order because phases can coexist, separated by a boundary with surface tension. (At the boundary "loops end" one ties up, at a cost in free energy proportional to area of the interface.)

**Dynamical Quarks**

So far we have discussed confinement and Higgs phases in $G = SU(N)/U(N)$. But what if we introduce matter in the fundamental representation (or some other rep that does not represent the center trivially?)

With dynamical $fN$ electric charges, electric flux tubes are unstable - the tube can break via nucleation of a pair of charges:
Dynamical changes

holes appear in the string world sheet, and expand --

\[ \langle W | \chi \rangle \propto e^{-\mu(\text{Perimeter})} \]

irrespective of whether there is confinement.

What about the 'throat loop'?

It is the world line of a classical \( Z_n \) monopole, or the boundary of a \( Z_n \) domain string.

But the fundamental irrep matter can see the string -- so the string is a physical object rather than a gauge artifact.

\( \mathcal{B}(\mathcal{C}, \Sigma) \) becomes \( \mathcal{B}(\mathcal{C}, \Sigma) \) -- it depends on how the world sheet of the distorting is chosen -- and the string has actual proportional to its area.

(Virtual \( Z_n \) charges nucleate and wind around the string before reannihilating)

\[ \langle \mathcal{B}(\mathcal{C}, \Sigma) \rangle \propto \exp \left( -\frac{1}{2} \text{Area}(\mathcal{B}) \right) \]

irrespective of whether there is confinement.
Nevertheless, it is possible to distinguish different realizations of the $\mathbb{Z}_N$ symmetry.

Recall the case of adjoint Higgs fields that completely Higgs $\text{SU}(N) \to \mathbb{Z}_N$

so there are stable $\mathbb{Z}_N$ magnetic flux tubes

Now suppose there are also Higgs fields that carry $\mathbb{Z}_N$ charge.

These particles have an infinite range Shnorr-Coleman interaction with the flux tubes. Hence, in principle a $\mathbb{Z}_N$ charge can be detected by a string that is infinitely far away and as we have argued previously the theory has $\mathbb{Z}_N$ charge superselection sectors.

What happens if the fundamental map Higgs field condenses? Then $\mathbb{Z}_N$ is spontaneously broken and the flux tube is unstable, it becomes the boundary of a domain wall.
Hence there is a phase boundary, separating phase with a $Z_N$ superselection rule from phase without one.

What is an order parameter for this transition? We can define an operator $F(\Sigma)$ ("flux operator") that inserts into the path integral a closed, inextensible string worldsheet on $\Sigma$ (gauge field is constrained so $e^{i\delta A} = e^{2\pi i n}$ on path $\Sigma$ that links with $\Sigma'$.) As already noted, field fluctuations see the string and

$$\langle F(\Sigma) \rangle \sim \exp(-J_{\text{Area}})$$

But consider

$$\langle F(\Sigma) W(\gamma) \rangle$$

where $\gamma$ links with $\Sigma$. If there is a charge superselection rule, this correlator has an Abelian-Bloch phase that is sensitive to the linking number of $\Sigma, \gamma$. We consider...
\[ A(\Sigma, C) = \frac{F(\Sigma) W(C)}{\langle F(\Sigma) \rangle \langle W(C) \rangle} \]

- The ABDP - or "Aharonov-Bohm order parameter" - can

\[ \lim \langle A(\Sigma, C) \rangle = \exp \left( \frac{2\pi i}{\hbar} K(\Sigma, C) \right) \]

in the "free-change" phase with \( Z_N \) superselection sectors

\( K(\Sigma, C) \) is the linking number of \( \Sigma, C \), and
the limit is taken so that \( \Sigma, C \),
though linked, are arbitrarily far
apart.

In the phase with no free changes

\[ \lim \langle A(\Sigma, C) \rangle = 1 \]

It seems like there are two distinguishable
phenomena that could destroy the
\( Z_N \) superselection rule. A Higgs
condensate with \( Z_N \) charge screens the
\( Z_N \) charge, and destabilizes the magnetic
flux tubes. Enforced confinement causes
\( Z_N \) charges to appear only in markets
where they are fixed to alternate charges
(or baryons, with trivial \( Z_N \) charge).
How do we distinguish these two?
we don't.

There appears to be no well-defined boundary that separates Higgs and confinement phases, even when there is matter transforming under \( Z_N \).

Example: \( G = SU(2) \) with a scalar doublet \( (\phi, \bar{\phi}) \) and a non-chiral fermion doublet \( (\psi, \bar{\psi}) \). Let's compare the spectrum of the theory in the case of Higgs behavior and confinement behavior to see if it is possible for the spectrum to evolve continuously from one regime to the other.
For $V^2 > 0$, we usually describe the spectrum of this theory in the following way. We use the gauge freedom to rotate $\phi$ so that

$$\phi_1 = 0$$
$$\phi_2 = V + \vec{\phi}.$$  

This is the unitary gauge, in which the spectrum contains no unphysical gauge artifacts.

In this gauge, the spectrum consists of

- scalar $\vec{\phi}$
- fermions $\psi_i, \psi_i^*$
- massive gauge bosons $W_1, W_2, W_3$

(potential composite states bound by the (weak) gauge interaction)

For $V > 0$, we expect that the theory is confining — physical states couple only to gauge-invariant (color singlet) operators. We classify the spectrum by these operators, which can create physical states when acting on the vacuum.

- scalar $\phi_i^0 \phi_i^0$
- fermions $\phi_i^0 \phi_i^*$
- $\epsilon_{ij} \phi_j \phi_i$
- vector bosons $\epsilon_{ij} \phi_i^0 \partial \mu \phi_j^0$
- $\epsilon_{ij} \phi_i^0 \phi_j^* \partial \mu \phi_j^* + \phi_i^0 \partial \mu \phi_i^0$

But only gauge-invariant operators couple to our states in the Higgs phase. If we impose the unitary gauge condition

$$\phi_i^0 \phi_i^0 \rightarrow V \vec{\phi}^0 +$$
$$\phi_i^0 \phi_i^* \rightarrow V \phi_i^0 +$$
$$\epsilon_{ij} \phi_i \phi_j \rightarrow V \phi_i +$$
\[ e_{ij} \Phi^i \otimes \bar{\Phi}^j \rightarrow v^2 W_{\mu}^{12} + \quad v^2 W_{\mu}^{3} + \quad \]

We find no reason to believe that the spectrum behaves discontinuously when \( v^2 \) changes sign.

One might imagine trying to distinguish the Higgs and confinement phases by a clever trick: couple the scalar and fermion to a U(1) gauge field, through the current

\[ J_{\mu} = \frac{1}{2} \left( \partial_{\mu} \Phi + \Phi \partial_{\mu} \bar{\Phi} + \Phi \bar{\Phi} \right) \]

Now the elementary scalar and fermion have charge \( e/2 \), but the gauge-invariant states have charge \( e \). Can we then tell whether the elementary particles are liberated by the Higgs mechanism?

No, because \( <\Phi> = (\frac{v}{\sqrt{2}}) \) breaks \( SU(2) \times U(1) \) down to a new \( U(1) \) with

\[ Q' = Q + T_3 \]

under which the elementary fields carry integer charges.

The expectations for the charges of the physical states are the same in the Higgs picture and the confinement picture. \( Q' = Q \) for a gauge-invariant operator, and \( v \) has \( Q' = 0 \).

Now contrast the above discussion with the case of a model with a scalar in the triplet representation of \( SU(2) \)

\[ (\Phi_1) \]

If there are no fermions in the defining rep, then we can define \( B/C \), and also expect \( \langle B/C \rangle \) to be a reasonable order parameter for confinement, because color electric \( Z \) charge cannot be screened.
Thus, it is possible to distinguish a Higgs phase from a confinement phase. And the distinction survives even when we introduce the fermion doublet \( \overline{q} \), as we see by considering the spectra.

In the Higgs phase, we expect \( SU(2) \to U(1) \) and we have, in unitary gauge:

\[
\begin{align*}
\phi_1 &= \Phi_1 = 0 \\
\phi_2 &= \nu + \bar{\nu}
\end{align*}
\]

Scalar \( \Phi \) (neutral)

Fermions \( \nu \), \( \bar{\nu} \) (charge \( \pm 1 \), unconfined)

Massive \( W^+ \)

Massless \( W^0 \)

And in the confinement phase, we have:

Scalars \( \phi_i \), \( \phi_i \overline{\phi_i} \)

Massive Vector \( \phi_i D^i \phi_i \)

Therefore, no fermions.

Because states with odd fermion number have nontrivial \( \mathbb{Z}_2 \) and physical states are all trivial under \( \mathbb{Z}_2 \). Apparently, there is a phase transition.

What is the fundamental difference between these two examples? Clearly, it is the existence of a surviving \( \mathbb{Z}_2 \) symmetry which can be used to classify states, when we say that the gauge symmetry is broken, we mean that there exist physical states which transform nontrivially under \( \mathbb{Z}_2 \). (In the \( SU(2) \) model with a Higgs triplet, \( \mathbb{Z}_2 \) happens to coincide with \( \mathbb{Z}_2 \) where \( F \) is fermion number.) We cannot make sense of this condition, however, if the unitary gauge order parameter transforms nontrivially under \( \mathbb{Z}_2 \). Then all states have indefinite \( \mathbb{Z}_2 \) quantum numbers.
Consider now the implications for the deconfining transition in the presence of dynamical quarks. The "pure gauge theory limit" is $m = \infty$, since for infinite mass quarks, breaking of the string can be ignored. For $m < \infty$, the $\mathbb{Z}_2$ global symmetry discussed on p. 748 is explicitly broken; quarks, transforming as the defining representation of $SU(N)$, satisfy boundary conditions which are not left-invariant by the $SU(N)/\mathbb{Z}_2$ gauge transformations considered here.

i) $N=2$

Neglecting dynamical quarks, we argued that large closed loops of electric flux would "condense" at a temperature $T = \sqrt{\frac{\mu}{\pi}}$. In this temperature range, the thermal estimate of the "tunneling" probability applies.

No matter how large $m$ is, breaking of the string will prevent arbitrarily large loops from forming. It seems reasonable to guess, then, that the 2nd-order phase transition will disappear for $m < \infty$. This is consistent with standard wisdom; 2nd-order phase transitions are typically unstable under small perturbations. (Eq. Ising magnet in external field)

ii) $N > 3$

For $m = \infty$, we argued that the deconfining transition, driven by percolation of a string network, was first order. For $m$ large, the time scale $\sim 2m/3$ for a link in the network to break is large compared to the characteristic time scale for fluctuations of the network. So a phase transition can still be expected. The order parameter $<\bar{e}iU>$, while nonzero in both phases, is discontinuous at some critical temp depending on $m$. 

$\beta C \rightarrow \infty$ since no symmetry distinguishes the two phases (flux tube screening and debye screening), it is possible for the transition line to terminate on

$\sim m/\sqrt{\pi}$