Physics 230

Elementary Particle Theory

Topic: Quantum Chromodynamics (QCD) - The Theory of the Strong Interactions

Course Outline:

1) The Renormalization Group and Perturbative QCD.
2) Quark Confinement.
3) Chiral Symmetry and Anomalies.
4) The Limit $N_c$ (number of colors) $\to \infty$.
5) QCD on the Lattice.

Prerequisite: A course in quantum field theory (e.g., Physics 205).

Text: Lecture Notes

Requirements: Problem Sets
Quantum chromodynamics is a great triumph for quantum field theory. It turned out that no fundamentally new principles were needed to understand the strong interactions. Instead, the old principles—

- Causality
- Unitarity
- Gauge invariance
- Renormalizability

were eventually found to have remarkable, qualitatively new consequences: e.g., asymptotic freedom and confinement.

Concepts which paved the way for QCD were—

i) The Quark Model:
   - Hadron resonances → Eightfold Way → Quarkmodel
   - $qq, q qq$ A classification of resonances—
   - $q qq$ No exotics, no isolated quarks.
   - But are quarks fundamental degrees of freedom? An affirmative hint came from current algebra.

ii) Color:
   - $SU(3)_c$ symmetry. Quark model "explained" if color singlets are much lighter than nonsinglets
   - $\bar{q}q > 1 \quad g, g g, \quad \bar{q} g, q \bar{g} \quad \bar{q} q g, q g \bar{q}$

   Also, statistics problem—
   - $\Delta^{1+} = uu u$ inconsistent with Fermi stat.
Later confirmation that no. of colors is $N_c = 3$

came from: $\pi^0 \rightarrow 2\pi$ $\propto N_c^2$
$\delta (e^+e^- \rightarrow \text{hadrons}) \propto N_c$

iii) Parton Model:

Electroproduction scaling -

$\sigma \sim \frac{1}{Q^4} f(\frac{Q^2}{2p_g})$

This is behavior expected if masses become irrelevant at large $Q^2$.
But it can occur only if strong interactions become weak at large $Q^2$ (short distances) - i.e.,
if there are no heavy resonances

$\Rightarrow$ we need asymptotic freedom, and the only renormalizable quantum field theories which are asymptotically free are non-abelian gauge theories.

We already need a color degree of freedom, so gauge it!

iv) ETC.:

The above considerations already lead us to an essentially unique theory of the strong interactions:

**QCD.** This theory better be able to explain:

- Quark confinement, and linear potential at large distances
  (Regge trajectories, quarkonia)
- Spontaneous breakdown of chiral symmetry
  (light pion)

...and, more quantitatively:

- Hadron spectrum, cross sections, structure functions and scaling violations,
B. Formulating QCD

We need to construct a theory of colored quarks, with local SU(3) symmetry:

\[ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \quad g(x) \rightarrow \Sigma(x) g(x) \quad \Sigma(x) \in SU(3) \]

To construct gauge-invariant kinetic term for quark, note

\[ \partial_{\mu} (\Sigma g) = (\partial_{\mu} \Sigma) g + \Sigma \partial_{\mu} g \]

so define

\[ D_{\mu} g = \partial_{\mu} g - ig A_{\mu} g \]

where \( A_{\mu}(x) \) is hermitian, traceless 3x3 matrix, and

\[ A_{\mu}(x) \rightarrow \Sigma A_{\mu} \Sigma^{-1} + \frac{ig}{2} (\partial_{\mu} \Sigma) \Sigma^{-1} \]

then \( D_{\mu} g \rightarrow \Sigma D_{\mu} g \) (transforms covariantly) and \( \overline{g} i D_{\mu} g \) is gauge-invariant

Choose a basis:

\[ A_{\mu} = A_{\mu}^a T^a \quad \epsilon^{abc} \delta_{ab} = \frac{1}{8} \delta_{abc} \]

Gluons are also dynamical. To construct kinetic term, note

\[ [D_{\mu}, D_{\nu}] g = \left\{ -ig (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) - ig^2 [A_{\mu}, A_{\nu}] \right\} g \]

Define

\[ F_{\mu\nu} = \frac{1}{ig} [D_{\mu}, D_{\nu}] = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig [A_{\mu}, A_{\nu}] \]

then \( F_{\mu\nu} \rightarrow \Sigma F_{\mu\nu} \Sigma^{-1} \)

\[ \Rightarrow \sigma F_{\mu\nu} F_{\mu\nu} \text{ is gauge-invariant} \]
Let's express this in terms of gluon fields $A^a_{\mu}$:

$$[T^a, T^b] = i C^{abc} \mathbf{T}^c$$

$$\Rightarrow C^{abc} = -2 i \, tr \, [T^a, T^b] \mathbf{T}^c$$

$c^{abc}$ is totally antisymmetric.

$$F_{\mu \nu}^a = 2 \, tr (F_{\mu \nu} T^a) = 2 \mathbf{A}_{\mu}^a - 2 \mathbf{A}_{\nu}^a + g C^{abc} A_{\mu}^b A_{\nu}^c$$

and

$$-i \, \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu} = -\frac{i}{4} F^a_{\mu \nu} F^a_{\rho \sigma}$$

conventionally normalized vector meson kinetic energy

The Lagrangian

Suppose there are $N$ quark flavors

$\bar{q}_n \, \gamma_{\mu} \, q_n \, n = 1, 2, \ldots, N$

Most general gauge-invariant, Lorentz-invariant, renormalizable Lagrangian is

$$\mathcal{L} = -\frac{1}{2} \, \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} + \bar{\mathbf{q}} L A L \mathbf{q} + \bar{\mathbf{q}} R B i \mathbf{q} R$$

$$- \bar{\mathbf{q}} L M L \mathbf{q} L + \bar{\mathbf{q}} R M R \mathbf{q} R + \frac{1}{2 \pi} \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}$$

$A, B, M$ are $N \times N$ matrices acting on flavor indices, $A, B$ are Hermitian, and

$$q_{L, R} = \frac{1}{2} (I + i Y_5) q$$

Now we perform some redefinitions to put $\mathcal{L}$ in a standard form:

1. First, we dispose of the last term:

**Exercise 1.1:** Show that

$$\epsilon_{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} = 2 \mathbf{K}_0$$

and find $\mathbf{K}_0$. 
This term is irrelevant in the classical theory; we may drop it. (But, we will need to reconsider this argument later, when we study the quantum theory.)

ii) \( A \mu \rightarrow \mathbb{Z} \frac{\hbar}{2} A \mu \)
\( g \rightarrow \mathbb{Z}^{-1} g \) — this gets rid of \( \mathbb{Z} \)

iii) \( A, B \) hermitian \( \Rightarrow \) they can be diagonalized by
\( A \rightarrow U_L^+ A U_L \), \( U_{L,R} \) unitary
\( B \rightarrow U_R^+ B U_R \)

Then rescale \( q_L \) in \( q_R \) to get
\[ \bar{q}_L i \tilde{\sigma}_L g_L + \bar{q}_R i \tilde{\sigma}_R g_R \]
\[ = \bar{q} i \tilde{\sigma} g \]

iv) Now we have masses
\[ \bar{q}_L M' q_R + q_R M' + q_L \]

But \( M' \) can be diagonalized by
\( M' \rightarrow V_L^+ M' V_R \), \( V_{L,R} \) unitary.
(without changing kinetic term)

Proof: \( M' \) is hermitian, positive.
\[ U^+ M' M' U = D^2 \]
\( U \) unitary
\( D \) real, diagonal, positive

If \( D \) has no zero eigenvalues —
\[ D = D^{-1} U^+ M' M' U \]
let \( U = V_R \)
\[ M' U D^{-1} = V_L \]
(\( \text{and} \) \( V_L^+ V_L = D^{-1} U^+ M' M' U D^{-1} = I \))

Exercise 1.2:
Complete the proof by disposing of the case in which \( D^2 \) has zero eigenvalues.
\[ L = -\frac{\lambda}{4} \text{tr} F^2 + \sum_n (\bar{\Psi}_n i D \Psi_n - m_n \bar{\Psi}_n \Psi_n) \]

\( g \) and \( k \) and \( m_n \) are the free parameters.

**Symmetries of the Lagrangian**

1. \( L \) is invariant under C, P, T
   
   (Recall P: \( \bar{\Psi} R \rightarrow \bar{\Psi} R \), \( \bar{\Psi} L \rightarrow -\bar{\Psi} L \)  
   C: \( \bar{\Psi} R \rightarrow \bar{\Psi} L \), \( \bar{\Psi} L \rightarrow -\bar{\Psi} R \))

2. Flavor symmetry
   
   \( \Psi_n \rightarrow e^{i\theta_n} \Psi_n \Rightarrow u, d, s, c, t, b \) conserved
   (charge, baryon number, strangeness)

3. Chiral Symmetry
   
   QCD interaction (and whole Lagrangian in \( M \rightarrow 0 \) limit) respects chiral symmetry
   
   \( U(1)_L \times U(1)_R \)
   
   \( \Psi_L \rightarrow V_L \Psi_L \)
   \( \Psi_R \rightarrow V_R \Psi_R \)

In case \( N=3 \), success of current algebra suggests that breakdown occurs:

\( SU(3)_L \times SU(3)_R \rightarrow SU(3)_V \)

(\( \pi, K, \eta \) are light)

(whether this actually occurs in QCD is a difficult dynamical question, to which we shall return)

Note, quark masses transform as

\( (3, \bar{3}) + (\bar{3}, 3) \rightarrow SU(3) \times SU(3)_R \)

or \( 8+1 \) of \( SU(3)_V \)

\( \Rightarrow \text{Gell-Mann-Okubo formula, among other things.} \)

- isospin
iv) Problem (U(1) Problem):

What about \( U(1)_A \)? There is no ninth coldstone boson, and if \( U(1)_A \) is not spontaneously broken, spectra should be in \( SU(3)_L \times SU(3)_R \times U(1)_A\times P \) representations, where \( P \) is parity.

Axial charge \( Q_A \) does not commute with parity, hence \( \mathbb{Z} \). If \( \mathbb{Z} = -1 \), then

\[
\mathbb{Z} \quad Q_A \quad 14 = + Q_A \quad 14
\]

spectrum parity-doubled.

We will say more about this later.

Note:

These symmetries are not completely destroyed by weak-interaction radiative corrections, even though they violate \( C, P, CP \).

Why?

These corrections generate operators of ...

- Dimension \( \leq 4 \) \( \Rightarrow \) absorbed into our most general \( \mathcal{L} \)
- Dimension \( > 4 \) \( \Rightarrow \) suppressed by powers of \( \frac{1}{M_W^2} \)

Things would have been different if we had chosen the strong interaction gauge group to be, e.g.,

\( SU(3)_L \times SU(3)_R \)

Then parity is conserved if \( g_L = g_R \) but not by the most general Lagrangian. Thus, weak radiative corrections will generate parity violation in the strong interaction in \( O(\alpha) \).

\[
\frac{1}{\mu^2 M_W^2} - \frac{1}{M_W^2} \quad \frac{1}{\mu^2} \quad \text{dim} = 4
\]

\[
\frac{1}{\mu^2} \quad \text{dim} = 2 \frac{1}{4}
\]
Aside:

The above remark helps to clarify the special status of renormalizable field theories, i.e.

- In general, operators of dimension 7+4 in the Lagrangian are expected to be suppressed by inverse powers of a large mass scale.

This is one reason why it was sensible to confine our attention to renormalizable theories when we constructed the QCD Lagrangian.

C. Interaction Vertices

What do the interaction terms in the QCD Lagrangian look like?

We have: \[ \frac{g}{2} \partial_{\mu} A^a \partial^\mu T^a \]

There are eight gluons;
six change color, two change color, but do not change it.

Since gluons carry color charge, they have self-couplings:

\[ k \left( \partial_{\mu} A^a - 2 A_{\mu}^a - i q [A_{\mu}^a, A_{\nu}^b] \right)^2 \]

contains

\[ g, \quad g^2 \]

A crucial difference between QED, QCD.

- Linear coupling
- Quartic coupling
One gluon exchange:

In the classical theory, what is the force between static quarks?

This is generated by the tree graph, and differs from the Coulomb potential only by a group-theoretic factor

\[ E(r) = \frac{g^2}{4\pi r} \left( \frac{1}{k_1} \right) (T^a)_{ij} (T^a)_{kj} \]

(sum on \( a \))

Now, write

\[ T^a_1 T^a_2 = \frac{1}{2} \left[ (T^a_1 + T^a_2)(T^a_1 + T^a_2) - T^a_1 T^a_1 - T^a_2 T^a_2 \right] \]

\[ T^a T^a = \epsilon_{\lambda} \Pi_{\lambda} \] acting on an irreducible representation of \( SU(3) \)

E.g., consider interaction between quark and antiquark

\[ 3 \times \bar{3} = 1 + 8 \]

Move is a singlet channel and an octet channel

\[ (T^a T^a)_{\text{singlet}} = \left( \frac{8}{3} \right) (\frac{1}{2}) = \frac{4}{3} \]

\[ (T^a T^a)_{\text{octet}} = 3 \]

Thus

\[ E(r) = \frac{g^2}{4\pi r} \left\{ \frac{-4}{3} \right\} - \text{singlet} \]

\[ \frac{4}{3} \]

\[ \frac{1}{2} \] - octet

9q attract in singlet channel

\[ 9q \] rope in octet channel

**Exercise 3**: Find interaction energy between two quarks in \( \bar{3} \) and \( 6 \) channels.
Addendum to 18:  
Gauge Field as a Connection

The gauge field can be regarded as a "connection" on a "fibre bundle", as was first pointed out by Hermann Weyl. This language means that the gauge field defines a notion of "parallel transport".

Suppose I wish to compare the colors of two quarks at different points in spacetime, e.g., to see if they are the same. In principle, I could do this by transporting one quark to the position of the other, and then comparing, but there is an ambiguity; really, two ambiguities. First, we must have a criterion for deciding that the color of the quark is not changed as it is transported; that is, a notion of "parallel transport". This notion is provided by the gauge field. Second, even for a given gauge field, the outcome of the comparison may depend on the path taken between the positions of the two quarks.

The gauge field is related to parallel transport in the following way: The covariant derivative $D_{\mu}g$ of a quark field $g$ is the change in $g$ relative to a parallel transported $g$. That is, $D_{\mu}g$ is the difference between the change in $g$ going from $x$ to $x+\epsilon$, and the corresponding change if $g$ were parallel transported.
The condition for parallel transport is therefore
\[ \delta m q = 0 \quad \text{or} \quad \delta m q = i A m q \]
(The coupling constant \( g \) has been absorbed in \( A m \) for convenience.) Integrating this equation gives
\[ q(x') = \left[ P \exp i \int_x^{x'} A m dx'^k \right] q(x) \]
where \( P \) is a "path-ordering" symbol, analogous to the Time ordering symbol in Dyson's formula,
\[ U(t, t_0) = \left[ T \exp \left[ -i H_2 \right] \right] U(t, t_0). \]

Now we know how to compare the colors of quarks at different points, but the comparison is not very interesting, because its outcome is changed by a gauge transformation. A mathematician would say that the comparison, or the matrix \( P \exp i \int A m dx'^k \), depends on the coordinates of the "bundle" \( x \) and not on its intrinsic geometry. The way to identify geometric properties is to determine how the color of a parallel transported quark depends on the path along which it is transported, or, equivalently, how the color changes when the quark is transported around a closed path.

For example, compare the two paths shown around infinitesimal square parallel transport along \( 0 \):
\[ q \rightarrow [1 + i e A m (x + e \mu)] [1 + i e A m (x)] \]
Parallel transport along $\gamma$:

$g \rightarrow [I + i e A_\mu(x + e\nu)] [I + i e A_\mu(x)] g$

Now, take difference:

$\left( [I + i e A_\nu + i e^2 \partial_\mu A_\nu] [I + i e A_\mu] - [I + i e A_\mu + i e^2 \partial_\mu A_\nu] [I + i e A_\nu] \right) g$

$= i e^2 (\partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]) g$

$= e^2 i F_{\mu \nu} g$

We conclude that the change in $g$ under parallel transport about area $e^2 \nu$ oriented in $\mu \nu$ plane is

$\delta_{\mu \nu} g = e^2 i F_{\mu \nu} g$

$F_{\mu \nu}$ depends on coordinates in only a trivial way; the orientation of our axes in the color space; its eigenvalues are gauge-invariant. $F_{\mu \nu}$ is obviously antisymmetric — interchanging $\mu$ and $\nu$ corresponds to rotating the square in the opposite sense. In analogy to Riemannian geometry, $F_{\mu \nu}$ can be called the "curvature" of the "bundle."

The similarity to general relativity is evident. A gravitational field is a connection which defines parallel transport for a gyroscope (or locally inertial frame) in spacetime. The Yang-Mills field is a connection which defines parallel transport for a colored mark in spacetime.

(In electromagnetism, that the gauge field behaves as a connection is dramatically illustrated by the Aharanov-Bohm experiment.)
Addendum to I.C.

The interaction energy of two static color sources in the one-gluon-exchange approx is proportional to the group theory factor

\[(T_1^a)_{ki} (T_2^a)_{lj} \] (sum on \(a\))

This interaction is invariant under simultaneous \(SU(3)\) color rotations of objects 1 and 2. To diagonalize it and find its eigenvalues, we thus go to the basis of irreducible \(SU(3)\) representations contained in \(R_1 \otimes R_2\).

where source 1 transforms as the irreducible representation (IR) \(R_1\), and 2 transforms as the IR \(R_2\).

The above remark is exactly analogous to the observation that the \(E1S\) coupling, familiar in atomic physics, is rotationally invariant and can therefore be diagonalized by transforming to the basis of \(IRs\) of the rotation group, generated by \(\hat{F} = L + S\).

As in the case of the \(E1S\) couplings, we may use the trick of writing

\[T_1^a T_2^a = \frac{1}{2} \left[ (T_1^a + T_2^a)(T_1^a + T_2^a) - T_1^a T_1^a - T_2^a T_2^a \right],\]

since the \(T_1^a, T_2^a\) are the generators of simultaneous color rotations of both objects. If \(T_R^a, a = 1, \ldots, 8\) are the generators of any irreducible representation, then

\[T_R^a T_R^a = \mathbb{C}(k) \mathbb{I}, \quad a \text{ multiple of } \mathbb{R} \text{ identity} \]
To see this, note first that $T^a T^a$ commutes with all the generators:

$$[T^a T^b, T^c] = T^a [T^b, T^c] + [T^a, T^c] T^b = i C^{abc} (T^a T^e + T^e T^a) = 0$$

because we have chosen the generators so that $C^{abc}$ is totally antisymmetric. Since $T^a T^a$ is hermitian and can be diagonalized, we see that all the eigenvectors of $T^a T^a$ must preserve the eigenvalues of $T^a T^a$; if $R$ is irreducible, $T^a T^a$ must be a multiple of the identity.

Acting on the representation $R \otimes R_1 \otimes R_2$, we have

$$T^a T^a = \frac{1}{2} \left[ C(R_1) - C(R) + C(R_2) \right]$$

so to complete the task of finding the eigenvalues of $T^a T^a$, we need only compute $C$ (the quadratic Casimir invariant) for the IRs $R, R_1, R_2$. To simplify the computation take the trace of both sides of

$$T^a T^a = C(R) I$$

to obtain

$$\text{Tr} (\text{# of generators}) T(R) = C(R) (\text{dim of rep})$$

where $T(R)$ is defined by

$$T^a T^b = T(R) C^{abc}$$

(According to the convention on page 1.3, then, $T(R) = \frac{1}{2}$ for the defining, or triplet, rep of $SU(3)$.) We now have

$$C(R) = \left[ \frac{\text{# of generators}}{(\text{dim of } R)} \right] T(R)$$

which is useful because it is easy to calculate $T(R)$ directly.
To calculate \( T(R) \), note that we may use any appropriately normalized \( SU(3) \) generator. It is convenient to use the generator which in the defining \((8,1,0)\) representation is
\[
T^3 = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Then we can find \( T(R) = R(T^3)^2 \) by decomposing any \( SU(3) \) representation \( R \) into the direct sum of irreducible representations of the \( SU(2) \) generated by \( T^3, T^8, T^7 \) and using
\[
T(R) = \sum_{\text{su(2) reps}} T(I5)
\]

For example, the 8 rep of \( SU(3) \) breaks up under \( SU(2) \) into
\[
8 = 3 \oplus \bar{3} - 1 \text{ of } SU(3)
\]
\[
\rightarrow (2+1) \otimes (2+1) - 1 = 3 + 2 + 1 + 1 \text{ of } SU(2)
\]
For the 3 rep of \( SU(2) \), the eigenvalues of \( T^3 \) are 1, 0, -1, so \( T(R) = 2 \), and we therefore have
\[
T(R=8) = 2 + \frac{1}{2} + \frac{1}{2} = 3
\]
\[
C(R=8) = \frac{8}{8} \cdot 3 = 3
\]

For the \( R=3 \) rep of \( SU(3) \) we have
\[
C(R=3) = \frac{8}{3} \cdot T(R=3) = \frac{4}{3}
\]

For \( R=6 \) of \( SU(3) \), we have
\[
6 = (3 \times 3) \text{ symmetric of } SU(3)
\]
\[
\rightarrow [(2+1) \otimes (2+1)]_{\text{sym}} = 3 + 2 + 1 \text{ of } SU(2)
\]
and therefore...
For \( R = 10 \) or \( SU(3) \), we have
\[
10 = 6 \otimes 3 - 8 \text{ of } SU(3)
\]
\[
\Rightarrow (3+2+1) \otimes (2+1) - (3+2+2+1) = 4+3+2+1 \text{ of } SU(2)
\]
\[
\begin{align*}
T(R = 10) &= 5 + 2 + \frac{1}{2} = 15_2 \\
C(R = 10) &= \frac{8}{10} \cdot \frac{15_2}{2} = 6
\end{align*}
\]

Now you know all the Casimirs which are needed in the exercises.

If you would like to know more about its representations of \( SU(3) \), two good books are:

H. Georgi, "Lie Algebras in Particle Physics"

D. Lichtenberg, "Unitary Symmetry and Elem. Particles"

In particular, you should probably learn how to find the irreducible representations contained in the direct product of two \( SU(2) \) copies, but that knowledge is not indispensable for this course.

As an (unofficial) exercise, you may wish to find \( C(R) \) for some other representations of \( SU(2) \).
Renormalization

Recall: Sackfield theory
\[ G^{(n)}(x_1, x_2, \ldots, x_n) = \left< 0 \right| \frac{1}{\mathcal{Z}} \prod_{i=1}^{n} \phi(x_i) \left| 0 \right> \]

\[ \mathcal{Z} = \sum \text{all Feynman diagrams with n extended lines} \]

Heisenberg picture field

\[ \text{Physical vacuum} \]

Connected: Systematic expansion in number of "loops" \( L \)

\[ \text{Diagram } \sim (\mathcal{L}^{L-1}) \]

(\( n \) loops = \( n \) momentum integrals)

Example:
\[ \mathcal{L} = \left( \frac{1}{2} m \phi^2 - \frac{1}{2} m^2 \phi^2 - \frac{i}{2} \lambda \phi^3 \right) / \hbar \]

Feynman rules
\[ p \to \frac{i}{\sqrt{-m^2 + i\epsilon}} (\text{momentum}) \]

\[ \lambda = -i \hbar (\text{momentum space}) \]

Connected \( \Rightarrow \)

\[ \mathcal{L} = \mathcal{I} - \mathcal{V} + \mathcal{I} \]

\[ \mathcal{I} = \int \delta^4(x - y) \frac{i}{p^2 - m^2 + i\epsilon} \]

\[ \mathcal{V} = \hbar^{L-1} \left( \text{factor}\right) \delta^4 (\text{total}) \]

Or: write field: \( \phi = \frac{1}{\sqrt{2}} \phi \)

\[ \mathcal{L} \rightarrow \frac{1}{\hbar^2} \mathcal{L} \]
Mass 1 Renormalization

Define \( \Pi \), \( \Lambda \), etc.

- cannot be disconnected by cutting a single internal line

\( \text{NO} \)

\( \text{YES} \)

\[ G^{(2)}(x, y) = \langle 0, \rho | \mathcal{O}_N(x) \mathcal{O}_N(y) | 0, \rho \rangle = \text{sum of Feynman diagrams} \]

\[ G^{(2)}_{\text{con}} = \text{connected} \]

\[ = \quad + \quad \Pi \quad + \quad \Pi \quad + \quad \]

\[ = \quad + \quad \Pi \quad + \quad \Pi \quad + \quad \]

\[ = \quad + \quad \Pi \quad + \quad \Pi \quad + \quad \]

\[ = \quad + \quad \Pi \quad + \quad \Pi \quad + \quad \]

\[ = \quad + \quad \Pi \quad + \quad \Pi \quad + \quad \]

\[ \text{Note} \quad \text{Diagram: \( m \rightarrow 0 \), \( \delta \rightarrow m \), \( \lambda \rightarrow \lambda \)} \]

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\[
\tau \sim \tau (m^2) + (p_{\text{out}}^2 - m^2) \tau (m^2) + \cdots
\]

\[\text{residue residue of pole}
\]

Significance of this?

E.g., use Green function to compute scattering amplitude (see Catan)

\[
\text{Con} = \begin{array}{c}
\text{IP1} \\
\text{IP2}
\end{array} + (2 \text{ crossed})
\]

\[\text{5-channel pole, determined by pole of exact propagator}
\]

\[
\text{Unitarity} \rightarrow \text{location of (analytically continued) amplitude is mass of physical particle}
\]

See this more directly (work with Green function instead of S matrix):

\[
\text{Con} = \begin{array}{c}
\text{O1} \\
\text{O2}
\end{array} + \begin{array}{c}
\text{O3} \\
\text{O4}
\end{array}
\]

\[\text{Land pole}
\]

(Here "vacuum bubbles" = multiplicative constant are excluded, remember)

\[
= \left< 01 \Gamma [\partial (x) \phi_{14}] 10 \right>
\]

\[- \left< 01 \phi (x) 10 \right> \left< 01 \phi_{14} 10 \right>
\]
Tadpole: \( \langle 0 | \phi(x) | 10 \rangle = \langle 0 | \phi(0) | 10 \rangle \) = constant

(Translation invariance of \( |10\rangle \))

(Feynman diagram \( \Delta^4(x) \))

Often convenient to "shift" field
\( \phi(x) = \langle 0 | \phi(0) | 10 \rangle + \tilde{\phi}(x) \)

So \( \langle 0 | \phi(x) | 10 \rangle = 0 \)

\( \langle 0 | \phi(x) | \bar{\phi}(y) | 10 \rangle \) (time ordering \( x^0 \geq y^0 \))

\[ = \frac{1}{\Lambda} \sum_i \frac{1}{(2 \pi)^3 2 \omega_k} \langle 0 | \phi(x) | 1k \rangle \langle k | \phi(x) | 10 \rangle \]

\[ = \langle 0 | \phi(x) | 10 \rangle \langle 0 | \phi(0) | 10 \rangle \] (Tadpole)

\[ + \frac{1}{\Lambda} \sum_i \frac{1}{(2 \pi)^3 2 \omega_k} \langle 0 | \phi(x) | 1k \rangle \langle k | \phi(y) | 10 \rangle \] (Rel. Norm 1 part. state)

\[ + \sum_i \langle 0 | \phi(x) | 1n \rangle \langle n | \phi(y) | 10 \rangle \] (2 or more particle state)

Trans. invariance:
\[ \langle 0 | \phi(x) | 1k \rangle = e^{-i k \cdot x} \langle 0 | \phi(0) | 1k \rangle \]

Lorentz invariance:
\[ \langle 0 | \phi(0) | 1k \rangle = \langle 0 | \phi(0) | 1\bar{k} \rangle \]

\[ = \sqrt{2} \] (\( k \)-independent constant)
\[
\mathcal{Z} \int \frac{d^3k}{(2\pi)^3} \sum_n e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \\
+ \sum_n e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \mathcal{Z} \langle 0 | \phi(n) | \mathbf{l} \mathbf{n} \rangle^2
\]

same as free-field theory

\[
\mathcal{Z} \Delta_+(\mathbf{x} - \mathbf{y}; m^2) + \text{Remainder}
\]

\[
\text{Remainder} = \int \frac{d^4k}{(2\pi)^4} \sum_n \delta(k \cdot l_n) \mathcal{Z} \langle 0 | \phi(0) | \mathbf{l} \mathbf{n} \rangle^2
\]

\[
= \int \frac{d^4k}{(2\pi)^4} e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \mathcal{Z} \langle 0 | \phi(0) | \mathbf{x} \mathbf{n} \rangle^2
\]

\[
= \int \frac{d^4k}{(2\pi)^4} e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \Theta(k^0) \mathcal{Z} \langle 0 | \phi(0) | \mathbf{x} \mathbf{n} \rangle^2
\]

Defined

\[
\mathcal{Z} \langle 0 | \phi(0) | \mathbf{x} \mathbf{n} \rangle^2 = \left( \frac{2\pi}{\mathcal{Z}} \right)^4 \sum_n \delta(k \cdot l_n) \mathcal{Z} \langle 0 | \phi(0) | \mathbf{l} \mathbf{n} \rangle^2
\]

(Note: Lorentz-invariant function)

Or

\[
\text{Remainder} = \int d\mu^2 \mathcal{Z} \langle \mu^2 \rangle \int \frac{d^4k}{(2\pi)^4} \Theta(k^0) \Delta_+(\mathbf{x} - \mathbf{y}; \mu^2) e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}
\]

\[
= \int d\mu^2 \Delta_+(\mathbf{x} - \mathbf{y}; \mu^2)
\]

We have

\[
\langle 0 | \phi(x) \phi(y) | 0 \rangle = \mathcal{Z} \Delta_+(\mathbf{x} - \mathbf{y}; m^2) + \int d\mu^2 \mathcal{Z} \langle \mu^2 \rangle \Delta_+(\mathbf{x} - \mathbf{y}; \mu^2)
\]

Lehmann–Källén spectral representation.

Superposition of free-field values.
combine time orderings

$$-\text{cut} = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int d\mu^2 \delta(\mu^2) \frac{1}{p^2 - \mu^2 + i\epsilon}$$

pole term

So -- o positive pole

= physical * mass

* residue at pole

$$Z = \langle 0 | \phi(0) | k \rangle k^2$$

-- overlap of $\phi(1_k)$ w/ physical vacuum

(2=1 in free theory)

Also spectral rep for commutator:

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$= Z i \Delta(x-y;m^2) + \int d\mu^2 \delta(\mu^2) i \Delta(x-y;m^2)$$

free theory

But -- canonical commutator

$$[\phi(x), \phi(y)] = i \delta^3(x-y)$$

$$1 = Z + \int d\mu^2 \delta(\mu^2)$$

$$= Z(1 + \int d\mu^2 \delta(\mu^2))$$
where \( \Phi = \sqrt{Z} \Phi \),
\[
\tilde{\sigma}(k^\nu) \Theta(k^\nu) = (2\pi)^3 \sum S(1) \hat{K}_0 \hat{B}(n) \xrightarrow{1}
\]
and \( \tilde{\sigma} \geq 0 \Rightarrow \boxed{0 \leq Z \leq 1} \)

(coupling of normalized field to many-particle states weaker coupling to one-particle states)

**Note:** can't find a field with

\[
\langle \Phi, \Phi \rangle = i \delta^3(-) \quad \int \langle \Phi | \Phi \rangle = 1
\]

(for \( \tilde{\sigma} > 0 \) — i.e., an interacting theory)

— Haag's Theorem

**Mass Renormal**: What have we obtained? Q. corrections modify ("renormalize") mass of particle

\[
m^2 \sim (\text{length})^{-2} \quad E = \frac{k}{\hbar} \sqrt{k^2 + m^2}
\]

i.e., \( \xi m = \text{mass, as } \lambda \to 0 \)

Denote

\[
\mathcal{Z} = \frac{1}{2} m \hat{\Phi} \hat{\Phi} - \frac{i}{2} m^2 \hat{\Phi}^2 - \frac{1}{3} \Phi^3
\]

\( m_0 = \) "classical mass" = "bare mass"

physical mass \( m \) found by solving

\[
p^2 - m_0^2 - \pi(p^2) = 0
\]
\[ \frac{1}{\pi} = \cdots + \text{higher order} \]

\[ M^2 = M_0^2 + O(\lambda^2) \]

Dimensionally = \[ M_0^2 f \left( \frac{\lambda^2}{M_0^2} \right) \]

\[ f(0) = 1 \]

Q. corrections to decay "the particles interacting with vacuum oscillators" change its mass

Reorganize the expansion

\[ \langle \pi, \pi \rangle \left( p_1 - p_n + 1, m_0^2 \right) \]

Find \( m^2 \) from pole in \( \cdots \)

\[ \text{Reorganize} \quad \langle \pi, \pi \rangle \left( p_1 - p_n + 1, m^2 \right) \]

More convenient: reparametrize

\[ M_0^2 = m^2 + \text{correction} \]

\[ \Rightarrow \quad \frac{i}{p^2 - m^2} \quad \frac{\text{physical}}{\text{mass}} \quad \text{term} \]

\[ \text{correction} = -i (\Delta m^2) \]

\[ \frac{\text{then}}{\text{then}} \quad \frac{\text{then}}{\text{then}} = \frac{i}{p^2 - m^2} - \frac{i}{p^2 - m_0^2} \]
Calculate $\Delta m^2$ order by order, by demanding

$$\Pi (p^2 = m^2) = 0$$

so pole in $m^2$ stays at $m^2$.

i.e.

$$\Delta \Pi = \Delta \Pi^{\text{1loop}} + \Delta \Pi^{\text{2loop}} - (\Pi^{\text{1loop}} + i)(\Delta m^2)^{\text{1loop}}$$

so

$$(\Delta m^2)^{\text{1loop}} = -\Pi^{\text{1loop}} (p^2 = m^2)$$

continue to insert counterterms iteratively

$$\Pi \rightarrow \Pi^{\text{2loop}} = \Pi^{\text{1loop}} + \Pi^{\text{2loop}} - 2 \Delta \Pi^{\text{2loop}}$$

$$= 2 \Pi^{\text{2loop}}$$

$$\text{or } 0(\Delta m^2)^{\text{1loop}}$$

$$\text{or } 0(\Delta m^2)^{\text{2loop}}$$

Calculating this way, we don't have to go to the trouble of solving for $m^2$ and reexpressing answer in terms of $\Delta m^2$.

If we want to know relation of $m^2$ to $m_0^2$, we have computed first, to, order-by-order, in the form

$$\Delta m^2 = m_0^2 - m^2 = m^2 f\left(\frac{1}{m^2}\right)$$

(i.e. in terms of $m^2$, instead of $m_0^2$ in terms of $m_0$)
Let's say we have performed mass renormalization, so
\[ \pi(p^2 = m^2) = 0 \]

Expand
\[ \pi(p^2) = (p^2 - m^2)\pi'(m^2) + \frac{1}{2}(p^2 - m^2)^2 \pi''(m^2) \]

That is
\[ \frac{i}{p^2 - m^2 - \pi(p^2)} \]
\[ = \frac{i}{p^2 - m^2 (1 - \pi'(m^2)) + O(p^2 - m^2)^2} \]

close to perturbative pole

\[ \sim \frac{iZ}{p^2 - m^2 + i\epsilon}, \text{ where } Z^{-1} = 1 - \pi'(m^2) \]

So we have calculated how \( \phi \) couples to one-particle states

\[ \langle 0 | \phi | 0 \rangle = \sqrt{Z} \]

where

\[ [\phi, \phi^\dagger] = i\delta^{(3)} \]

However, it is more convenient to calculate with renormalized fields

\[ \phi_R = Z^{-\frac{1}{2}} \phi \]

so that

\[ \langle 0 | \phi_R | 0 \rangle = 1 \]

(although

\[ [\phi_R, \phi_R^\dagger] = Z^{-1} i\delta^{(3)} \]
Mac invariant because was directly related to scattering amplitudes (see "LSZ reduction formula", below).

We can reorganize Feynman diagram expansion to that we calculate "renormalized" Green function directly

\[ G_{\phi}^{(n)}(x_1 - x_n) = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | \bar{0} \rangle = Z^{-n/2} G_{\phi}^{(n)}(x_1 - x_n) \]

Rewrite:

\[ Z = \frac{i}{2} (\partial_{\phi} \bar{\phi}) - \frac{1}{2} m_0^2 \phi^2 - \frac{i}{3!} \lambda \phi^3 \]

in terms of \( \phi_R, m \)

\[ \phi_B = Z^{1/2} \phi_R \quad \Rightarrow \quad m_0^2 = m^2 + \text{Im} \]

\[ Z = \frac{i}{2} Z (\partial_{\phi} \phi_R)^2 - \frac{1}{2} (m^2 + \text{Im}) Z \phi_R^2 - \frac{i}{3!} \lambda Z^2 \phi_R^3 \]

\[ = \frac{i}{2} (\partial_{\phi} \phi_R)^2 - \frac{1}{2} m^2 \phi_R^2 - \frac{i}{3!} \lambda \phi_R^3 \]

\[ + (Z - 2) \left[ \frac{i}{2} (\partial_{\phi} \phi_R)^2 - \frac{1}{2} m^2 \phi_R^2 \right] - \frac{i}{2} (Z \text{Im}) \phi_R^2 \]

New Feynman Rules:

\[ = -i Z S \text{Im} + i(Z - 2) (p^2 - m^2) \]

(derivative interaction)
We impose conditions
\[ \begin{align*}
\Pi(p^2 = m^2) &= 0 \\
\Pi'(p^2 = m^2) &= 0
\end{align*} \]
determine \( \Sigma \) and \( \Pi \) order-by-order

Assume that \( m^2 \) is a physical mass
\( \phi \) \( \square \) is renormalized field

**COUPLING RENORMALIZATION**

We calculate in terms of \( m^2 \), rather than \( m_0^2 \), because \( m_0 \) cannot be measured.

Similarly, convenient to express Green functions, not in terms of \( \phi \), but in terms of a more directly measurable quantity

Define
\[ -i \Gamma(p_1, p_2, p_3) \equiv \begin{array}{c}
p_1 \cr \hline
p_2 \end{array} \rightarrow \begin{array}{c}
p_3 \cr \hline
p_2 \end{array} \]

Let this invariant, and \( p_1 + p_2 + p_3 = 0 \),

So e.g. \( (p_1 + p_2) \Gamma = p_1 \Gamma + p_2 \Gamma + 2 p_1 \Gamma = p_3 \Gamma \rightarrow p_1, p_2 \) not independent invariant

(Rather arbitrary)

renormalization condition:

"On shell" \( \Gamma(m^2, m^2, m^2) = \Lambda R \)

Note: momenta must be nonphysical (not kinematically allowed for real timeline 4 momenta)

Defined by analytic continuation -- How can we measure it?
E.g.

\[ \begin{array}{c}
\text{(diagram)} \\
\text{[diagram]} \\
\text{(diagram)} \\
\text{[diagram]}
\end{array} \]

\[ \rightarrow \quad \text{(diagram)} \quad \text{(diagram)} + \text{etc} \]

\[ \sim \frac{(-i\lambda R)^2}{S-m^2} \quad \text{-- Near K-vector} \]

So -- residue (analytically continued) amplitude at \( p \rightarrow 0 \) on \( S \) related to nonrenormalized coupling.

\[ \begin{array}{c}
\text{(diagram)} = \text{(diagram)} + \text{(diagram)} + \text{h.o.} \\
\text{[diagram]} \\
\text{[diagram]} \\
\text{[diagram]}
\end{array} \]

we write

\[ Z_{\text{int}} = -\frac{1}{3} \lambda_0 \bar{\phi}^3 = -\frac{1}{3} \lambda_0 Z^{3/2} \bar{\phi}^3 \]

\[ Z^{3/2} \lambda_0 = \lambda R + \mathcal{O}(1) \]

-- Of course renormalization
the full QM coupling is corrected order by order up to -- we can calculate

\[ \lambda_0 = \lambda R + \mathcal{O}(\lambda R^3) \]

\[ = \lambda R \tilde{g}(\lambda^2/m^2) \]

\[ \tilde{g}(0) = 1 \]
For completeness: The other renorm conditions...

1. \( \langle P \rangle = 0 \) then we don't have to worry about e.g.

\[ \phi \frac{}{} \]

reflects our freedom to "shift" the field by a constant, so that \( \langle \phi \rangle = 0 \) c.t.

2. \( \langle P \rangle = 0 \)

two subtests vacuum energy

\[ \langle 0 \text{ vac} | 0 \text{ vac} \rangle = \langle 0 \text{ vac} | e^{-iH} | 0 \text{ vac} \rangle \]

\[ = e^{-i E_{\text{vac}}} \]

\[ = \exp \left[ \frac{\omega_0}{1} \right] \]

\[ = \exp \left[ \frac{c}{1} \right] \]

3. (wants to subtract Energ away)

(We are free to do this -- unless we couple K\(\text{eay}\) to gravity!)

**Summary:**

1. \( \langle P \rangle = 0 \)
2. \( \langle P \rangle = 0 \)
3. \( \langle P \rangle = 0 \) \( (p^2 - m^2) \)
4. \( \langle P \rangle \mid = -i \frac{d}{dR} \)

All improved order by order.
Infinite Mass Renormalization

Let us now consider, in our model with $N = \frac{1}{2} (-i \phi^3)$, the leading contribution to the quantity

\[ -i \Pi(p^2) = -p - \left[ i p^2 - p - \text{p + higher order} \right] \]

where

\[ F(x) = \frac{1}{2} (i \lambda x) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k+p)^2 - m^2 + i\epsilon} \]

(symmetric factor)

We can immediately see that this integral is divergent. For large $k$, it behaves like

\[ \int \frac{d^4k}{k^4} \]

So it will proportional to the log of the ultraviolet cutoff:

\[ \textcircled{C} \sim \ln \Lambda + \text{finite part} \]

We say that the ultraviolet divergence is "logarithmic".

We can also easily see, without doing an explicit calculation, that the "infinite" part of this diagram (the piece proportional to $\ln \Lambda$) is independent of the external momentum $p$. To see this, differentiate the graph with respect to external momentum:

\[ \frac{\partial}{\partial p^2} \textcircled{C} = \frac{1}{2} (i \lambda x) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k+p)^2 - m^2 + i\epsilon}^2 \]

Now counting powers of $k$, shows that the integral converges. This shows that if the graph is expanded in powers of $p^2$ (about any value of $p^2$) in a Taylor series, only the constant term is divergent.
(proportional to \( \ln \Lambda \)).

Therefore, without explicitly calculating, we know that

\[
\Pi(p^2) = \Lambda^2 \left[ C \ln \Lambda + \text{"finite part"} \right] + \text{higher order}
\]

where \( C \) is a constant, independent of \( p^2 \).

(The argument of the log should be dimensionless, so the finite part must include a term proportional to \( \ln \Lambda \). Also, since the integral is dimensionless, \( C \) must be a numerical constant, independent of \( \mu \).

Now suppose we perform mass renormalization, as described previously. That is, we include an \( \Pi(p^2) \) contribution from the mass counterterm:

\[
-i \Pi = \text{mass counterterm}
\]

The mass counterterm is chosen so that \( \Pi(p^2) = 0 \) and thus we see that

\[
\delta m^2 = -\Lambda^2 \ln(\Lambda/m) + \text{finite part} + \text{higher order}
\]

The infinity (the \( \Lambda \) dependence) gets completely absorbed into the relation between the bare and physical masses,

\[
\mu_0^2 = m^2 + \delta m^2 = m^2 - \Lambda^2 \ln(\Lambda/m) + \cdots
\]

and the expression for \( \Pi \) as a function of the physical mass \( m^2 \) is finite.

(Furthermore, since \( \Pi(p^2) \) is finite, we see that the wave function renormalization is finite -- i.e. \( Z \) has no dependence on \( \Lambda \).)
What is the meaning of this logarithmic divergence? It occurs because for $k^2 > m^2$, the quantum fluctuations have no intrinsic scale. If $k$ is small, the quantum corrections are in a sense small, but fluctuations on all scales of length contribute. For each factor of two, say, in distance scale, the contribution is small if $k$ is small, but because we get the same contribution from each factor of 2 in distance scale, the total contribution from fluctuations on all scales is infinite.

Evidently, then, even the low energy ($p^2 < m^2$) physics of our model is sensitive to the fluctuations at arbitrarily small wavelengths. This sounds like bad news. Because it may appear that, in order to predict low energy physics, we need to understand physics at arbitrarily short distances. But we can never expect any model field theory to provide an accurate description of physics at arbitrarily short distances. (QED, for example, is not a valid description above 100 GeV, let alone 1019 GeV!) Since we can't really know what the physics is at very short distances, sensitivity to very short wavelengths appears to mean loss of predictive power.

But the procedure described above indicates that this need not be so! When we perform mass renormalization, we obtain expressions for amplitudes in terms of the physical mass, and these expressions are insensitive to the short-wavelength physics. If we wish to relate different measured "low energy" quantities, no dependence on $m$ appears. All the sensitivity to short wavelengths can be isolated in the dependence of the physical parameters.
on the parameters in the "Hamiltonian of the World," e.g., the dependence of the physical mass on the bare mass.

That all of our ignorance about short-wavelength physics can be absorbed into the relation between bare and renormalized parameters (and isolated from the predictions of relations among measurable quantities) is the crucial idea in the theory of renormalization. Indeed, it might even be regarded as the control concept of (relativistic) quantum field theory, because it is only when we grasp this concept that we recognize that we need not be so arrogant as to suppose that we understand physics at the Planck scale, in order to understand physics at 100 GeV.

**Higher Orders**

The "ultraviolet behavior" of our $\frac{3}{2} + \phi^3$ model is actually quite simple, because the sensitivity to short wavelengths becomes milder and milder in higher orders of perturbation theory in $\lambda$.

For example, consider the next order contribution to $\Pi$:

\[-\Pi \Big/ \sim \frac{1}{\text{order}^4} = \text{term} + \text{term} + \text{term} + \text{term} + \text{term}\]

The behavior of the loop integrations in $\text{term}$ for large loop momenta is evidently

$$\int d^4k_1 d^4k_2 \frac{1}{k_1^4} \frac{1}{k_2^4} \frac{1}{(k_1 + k_2)^2}.$$
and we can see by counting powers of $k$ that this is convergent for large $k$. So the only cutoff dependence in $\pi$ in order $1^{-4}$ comes from --

![Circle diagram]

but we have already taken care of this with the order $1^{-2}$ mass counterterm. There is no new contribution to the infinite part of mass renormalization in order $1^{-4}$.

This situation persists in higher orders. Consider a general diagram with

- $L$ loops
- $I$ internal lines
- $E$ external lines
- $V$ vertices

The dimensionality of the loop integration (called the "superficial degree of divergence" of the diagram) is

$$ D = 4L - 2I $$

But recall the topological identities:

$$ L = I - V + 1 $$ (The number of integrations is the no. of propagators minus $V$ for the momentum-conserving $S$-function at each vertex, plus 1, since the overall momentum conserving $S$-function can be factored out of the graph.)

$$ E + 2I = 3V $$ (Both sides give the total no. of ends of lines kept as absorbed by vertices.)
Therefore, \[ D = 4(I-V+1)-2I = 2I - 4V + 4 = 3V - E - 4V + 4 \]
or \[ D = 4 - E - V \]

The number of superficially divergent diagrams is small for \( D > 0 \), we need \( E = 0, 1, 2, 3 \), \( V = 4, 3, 2, 1 \)

Vacuum diagrams are divergent up to order \( \lambda^4 \), but the only superficially divergent diagrams with external lines are \( \Phi, \Phi, \Xi \)

In particular, \( \Xi \) is (superficially) finite

-- for all the infinities must come from graphs like \( \Phi \)

These infinities are removed by an insertion of the order \( \lambda^2 \) mass counterterm.

Thus, this theory has no infinite coupling (or field) renormalization. The only infinite renormalizations are vacuum energy renormalization in order \( \lambda^2 \) and \( \lambda^4 \), mass renormalization in order \( \lambda^3 \), and the renormalization of \( \Phi \) in order \( \lambda \) and \( \lambda^3 \).

(More about this - "tadpole" later.)
More general:

\[ \text{coupling } \propto (\text{mass})^{\text{positive}} \rightarrow \overline{\text{superrenormalizable}} \]

Finite no. of "primitve divergences", so infinite renormalizations cease to be necessary after some finite order

(infinite counterterm = polynomial in coupling constant,

more interesting — "renormalizable"

\[ \text{coupling } = \text{ dimensionless} \]

E.g. consider \( \int \mathcal{L} = -\int d^d x \, A \phi^n \) 

\( n \) is integer

\[ \int d^d x (\partial \phi)^2 = \text{dimensionless } \Rightarrow \]

\( \phi \sim (\text{mass})^{(d-2)/2} \)

\[ \int (\text{mass})^{d - \frac{n}{2}(d-2)} = \text{dimensionless } \Rightarrow \]

\( \lambda \sim (\text{mass})^{d - \frac{n}{2}(d-2)} \)

\[ D = d - \left( \frac{d-2}{2} \right) E = \left[ d - n \left( \frac{d-2}{2} \right) \right] V \]

(Dimensional or topological analysis — p. 141)

\( E = 2 \Rightarrow \text{dimension (mass)}^2 \)

\( E = n \Rightarrow \text{dimension (A)} \)
We see from this argument that the reason for the mild ultraviolet behavior of this theory is that the coupling constant $\alpha$ has the dimensions of mass to a positive power. If we add a loop to a Feynman diagram, keeping the number of external lines fixed, we must add two vertices (since $V = 2L + E - 2$). Dimensional analysis thus shows that adding a loop decreases the dimension of the Feynman integral, and correspondingly reduces its degree of divergence.

In general, in a theory in which the coupling constant has positive dimension, there are a finite number of "primitive" divergences, and hence infinite renormalizations cease to be necessary after some finite order of perturbation theory. Such theories are said to be "superrenormalizable".

More interesting are the renormalizable theories, in which the coupling constant is dimensionless. In renormalizable theories, the primitive degree of divergence of a graph with a fixed number of external lines remains the same to all orders of perturbation theory, and thus the need for infinite renormalization persists to all orders.

Examples of renormalizable theories are scalar field theories with the interactions...

\[ H' = \frac{1}{3!} \phi^3 \quad \text{in 6 dimensions (dim } \phi = 2) \]
\[ H' = \frac{1}{4!} \phi^4 \quad \text{in 4 dimensions (dim } \phi = 4) \]
\[ H' = \frac{1}{6!} \phi^6 \quad \text{in 3 dimensions (dim } \phi = \frac{5}{2}) \]
(Any polynomial interaction in $\phi$ is superrenormalizable in 2 dimensions, since $\phi$ is dimensionless.)

For example, in the four-dimensional theory with $H = \frac{1}{4!} \phi^4$, the graph

$$\square \sim \lambda^2 \ln \Lambda + \cdots$$

is logarithmically divergent, and hence generates (logarithmically) infinite coupling renormalization in order $\lambda^2$. The graph

$$\bigcirc \sim \lambda^2 \left( \lambda^2 + \rho^2 - m^2 \right) \ln \Lambda + \cdots$$

is quadratically divergent.

It generates (quadratically) divergent mass renormalization, and (logarithmically) infinite field renormalization.

Dimensional analysis shows that coupling and field renormalization are always logarithmic, and mass renormalization is always quadratic in a renormalizable (scalar) field theory. E.g., consider the 6-dimensional theory with $H = \frac{1}{4!} \phi^5$:

$$\triangle \sim \text{logarithmically divergent}$$

$$\bigotimes \sim \text{quadratically divergent}$$

A crucial feature of renormalizable and superrenormalizable theories is that primitive divergences occur only in graphs with some small number of external lines. This feature allows us to isolate the effects of short-wavelength
quantum fluctuations in a few incoincident renormalizable parameters. This nice feature is not shared by theories in which the coupling constant has negative dimension, the so-called "nonrenormalizable" theories. In these theories, the ultraviolet divergence properties of Feynman diagrams get worse and worse in each order of perturbation theory.

For example, consider a 4-dimensional theory with

$$\mathcal{H} = \frac{3}{6!} \phi^6$$

we find that the diagram is quantum-mechanically divergent. To remove this sensitivity to short-wavelengths, we need a $\phi^8$ counter-term (since the graph has 8 external lines). Our interaction has become

$$\mathcal{H} = \frac{16}{6!} \phi^6 + \frac{16}{8!} \phi^8$$

where now there are two free parameters 16 and 16. But we find that

is now divergent.

To absorb this cutoff dependence into the relation between a bare and renormalized coupling, we must introduce a $\phi^{10}$ coupling. It is evident that we can absorb all the sensitivity to short-wavelength physics into our renormalized parameters only at the cost of introducing an infinite number of parameters. Thus, the theory has lost its predictive power. In nonrenormalizable theories, sensitivity to short wavelengths does destroy our ability to predict low energy physics.
So, in renormalizable and super-renormalizable theories, all the sensitivity to short-wavelength physics can be absorbed into a few parameters, which we can take to be the free parameters of the theory. And dimensional analysis determines what these parameters are. (In a renormalizable theory, they correspond to all the local terms in fields and derivatives of fields that have dimension less than or equal to $D$, in $D$ spacetime dimensions.) In non-renormalizable theories this procedure fails.

But, why are we so lucky that the world happens to be well described by a renormalizable field theory (like QED)? Or did it have to be this way?

In fact, there is a very good general reason to believe that low energy physics can be described to excellent accuracy by a renormalizable field theory. To understand this, let us take the idea of a cutoff very seriously. That is, we will accept the idea that our scalar field theory is the more low energy phenomenology of some more fundamental underlying field theory. The description of physics in terms of the Lagrangian

$$L(\phi(x), \partial \phi(x), ...)$$

is only approximate, and becomes inappropriate at wavelengths much smaller than $\Lambda^2$, where $\Lambda$ is the cutoff. This could happen for various reasons. Perhaps there are new elementary particles, with masses of order $\Lambda$. Or perhaps our scalar particles are not really elementary, but have a size of order $\Lambda^{-1}$, so that physics at shorter distances must be described in terms of their constituents.
When we adopt this point of view, there is no reason to expect the "phenomenological" lagrangian to be particularly simple. It ought to be local, Poincare-invariant, and should respect whatever other exact symmetries the theory has, but it need not be a polynomial in $\phi$ and derivatives, and could depend on derivatives higher than the first. Such a lagrangian, in a four-dimensional theory of a real scalar field, when expanded in powers of $\phi$, might have the form

$$L = \frac{i}{2} \partial \phi \partial \phi \phi - \frac{i}{2} m_0^2 \phi^2 - \frac{i}{4!} \lambda_1 \phi^4 - \frac{1}{6!} \lambda_6 \phi^6$$

$$- \frac{1}{4} \lambda_4 (\partial \phi \partial \phi \phi) \phi^2 + \cdots$$

(assuming a $\phi \to -\phi$ symmetry).

This is the bare lagrangian of a theory with cutoff $\Lambda$; it is very complicated.

But now we come to a crucial point: the coefficients of the operators in $L$ not of dimension 4, have dimensions of mass to a negative power. Since these coefficients are determined by physics at mass scale $\Lambda$ and above, dimensional analysis would indicate that, for example,

$$\lambda_6 = \lambda_6 \Lambda^{-2}$$

where $\lambda_6$ is a dimensionless number expected to be of order one (it has no reason to be very large).

Now, imagine that we compute Feynman graphs using $L$, with the understanding that all loop integrations are cut off at $k \Lambda$, and all external momenta $p$ obey $p^2 \ll \Lambda^2$.
We can see, just as a consequence of dimensional analysis, that the operators of dimension greater than 4 in $Z_4$ give a small contribution to the graphs, a contribution suppressed by powers of $p^2/\Lambda^2$. For example, consider graphs with 6 external lines:

\[ \sim 16/p^2 \]

\[ \sim 16/\Lambda^2 \quad --\text{suppressed by } p^2/\Lambda^2 \]

Similarly:

\[ \sim 16 \sum \frac{d^4 k}{(k^2+p^2)^3} \sim 16/p^2 \quad (\text{ignoring \ more\ terms}) \]

\[ \sim \frac{16}{\Lambda^2} \ln \frac{1}{p^2} \quad --\text{suppressed by } p^2/\Lambda^2 \]

Adding more 6 interactions makes graphs more divergent, but the extra powers of $1/\Lambda^2$ from the loop integrations are compensated by extra powers of $p^2/\Lambda^2$ from the coupling. E.g.

\[ \sim \left(\frac{16}{\Lambda^2}\right)^3 \left(\frac{A^4}{k^2}\right)^4 \frac{1}{(k^2)^6} \sim \frac{16}{\Lambda^2} \]

The dimension greater than four couplings are said to be "irrelevant" in the infrared. Their effects are suppressed by powers of $p^2/\Lambda^2$ relative to those of the renormalizable couplings.

There is an exception to this observation though. Namely, consider Feynman graphs with four or fewer external lines:
\[
\begin{align*}
\lambda & \sim 14 \left(\frac{\alpha}{(\kappa r^2)^2}\right)^2 \sim 16 \ln \left(\frac{\lambda^2}{\mu^2}\right) \\
\alpha & \sim \left(\frac{16}{\kappa^2}\right)^2 \left(\frac{\alpha}{(\kappa r^2)^3}\right)^4 \sim 16
\end{align*}
\]

--- same order, up to a logarithm

But this effect of the "irrelevant" couplings, which is not suppressed by powers of \(p^2/\mu^2\), can simply be absorbed into the definition of the renormalized coupling \((\lambda(x))\).

Thus, up to an accuracy of order \(p^2/\mu^2\), our very complicated bare theory can be replaced by a renormalized theory, with just one renormalized coupling and a renormalized mass as free parameters.

The relation between the bare Lagrangian and the renormalized Lagrangian is extremely complicated. But if we are interested in low energy physics at \(p^2/\mu^2 < 1\), we may compute to good accuracy using a simple renormalizable field theory (the most general renormalizable theory of one scalar field, with \(\phi - \phi\) symmetry).

The shift in viewpoint leads to the conclusion that non-renormalizable theories are not without predictive power after all. Worse about them is when we take the idea of a physical short distance cutoff seriously. For we then recognize that the coupling constants should be functions of the cutoff, and scale roughly as indicated by dimensional analysis. The essential physics is just a "decoupling" of long-wavelength physics from complicated details of short-wavelength physics.
short-wavelength physics is.

This new viewpoint is very useful in condensed matter physics as well as relativistic quantum field theory. It indicates that the long-wavelength behavior of any system should have a simple description, however complicated the microscopic physics. This idea is the key to the modern theory of second-order phase transitions, for instance, as developed by Ken Wilson. (The idea is called "Universality").

And it is, perhaps, THE MOST IMPORTANT IDEA IN PARTICLE PHYSICS. Because it enables us to understand why renormalizable quantum field theories work as a description of Nature, even though we may have no idea how things really are at extremely short distances. (And it explains, for example, why we can use QED at energies well below 100 GeV, even though to describe things accurately at energies of order 100 GeV, we need the Weinberg-Salam model.)

You might have noticed one implication of the above analysis. We ought to regard the mass term in the bare Lagrangian as just another coupling, and write

\[ m_0^2 = \Lambda^2 \lambda \]

The same logic as above indicates that \( \lambda \) is order one. And we know we expect that the bare mass of our scalar is of order \( \Lambda^2 \).
This is right, of course. \( \Lambda^2 \) will be gradually divergent, by dimensional analysis. We need a more counter-term of order \( \Lambda^2 \).

But, in the phenomenological Lagrangian point of view, we are given a bare theory at scale \( \Lambda \), determined by some underlying microscopic physics, and this bare theory determines \( \Lambda \) physical mass in turn. Why should it turn out that

\[
(\mu^2)_{\text{physicist}} \ll \Lambda^2
\]

This seems to require an incredible conspiracy among all the bare couplings of the free theory. Putting it differently, it requires that the bare mass be "fine-tuned" to an accuracy of \( \frac{\mu^2_{\text{physicist}}}{\Lambda^2} \), since

\[
\mu^2_{\text{physicist}} = \mu_0^2 + O(1/\Lambda^2)
\]

is small by virtue of a cancellation between two quantities of order \( \Lambda^2 \).

(This is a genuine problem. One says (beginning with Ken Wilson) that

"Elementary scalars are unnatural."

A scalar particle wants to acquire a mass of order the cutoff. Things are different with fermions and vector mesons, because new molecules can be forbidden by symmetries. There is, in fact, a symmetry called "supersymmetry" that can require that a scalar be massless, and our best reason for believing that supersymmetry has something to do with low energy (\(\lesssim 100\ TeV\)) physics is that an elementary ("Higgs") scalar seems to be required in the
Weinberg-Salam model. In any case, the need for an elementary scalar in the model encourages the hope that there is new physics -- a new cutoff -- not far above 1 TeV. So -- maybe the SSC will not be a waste of money.)
Infinite Coupling Renormalization

Consider \( \mathcal{Z} = \frac{1}{4!} \phi^4 \), \( d=4 \) dimensions (renormalizable)

\[
(\phi^4) = \langle \chi^4 \rangle + \text{counterterm} + \text{higher order}
\]

\[
\left. \left\langle \chi^4 \right\rangle \right|_{\text{symmetry factor}} = \frac{1}{2} (-i \lambda)^2 \sum \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k.p)^2 - m^2 + i\epsilon}
\]

"Feynman Trick"

\[
S_0' \left( ax + b(1-x) \right)^2
\]

\[
= \left( \frac{-i}{a-b} \right) \frac{1}{ax + b(1-x)} \bigg|_0^1
\]

\[
= \left( \frac{-i}{a-b} \right) \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab}
\]

\[
\langle \chi^4 \rangle = \frac{1}{2} \lambda^2 \int_{0}^{1} dx \int \frac{d^4 k}{(2\pi)^4} \left[ \frac{1}{(k^2 + 2k.p + x p^2 - m^2 + i\epsilon)^2} \right]
\]

And \( \left[ \right] = (k + x p)^2 + x(1-x)p^2 - m^2 + i\epsilon \)

We can shift \( k \) integration:

\[
\langle \chi^4 \rangle = \frac{1}{2} \lambda^2 \int_{0}^{1} dx \left( ax \right)
\]

\[
\left( ax \right) = \frac{1}{2} \lambda^2 \sum \frac{d^4 k}{(2\pi)^4} \left[ \frac{1}{(k^2 - a\epsilon + i\epsilon)^2} \right]^2
\]

"Wick rotate" the \( k^0 \) integral

Singularities at \( (k^0)^2 = k + a \pm i\epsilon \)
Either way, we can rotate $k^0$ contour

\[ I(a) = i \int \frac{d^4 k E}{(2\pi)^4} \frac{1}{[k^2 - \alpha + i\epsilon]^{7/2}} \]

- Drop $\epsilon$ subscript
- No longer need $i\epsilon$?
  We do need it, still, summed a become negative.

Rotational Invariance:

\[ d^d k = \Omega_{d-1} k^{d-2} dk \]

\[ \int d^d k e^{-K^2} = \pi^{d/2} \]

\[ \Omega_{d-1} \int_0^\infty dk k^{d-1} e^{-k^2} = \frac{1}{2} \Omega_{d-1} \int_0^\infty dk k^{d-1} (k^2)^{\frac{d-3}{2}} e^{-k^2} \]

\[ = \frac{1}{2} \Omega_{d-1} \Gamma\left(\frac{d}{2}\right) \]

\[ \Rightarrow \quad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \]

Integrate by parts

\[ \Gamma(z+1) = z \Gamma(z) \]

\[ \Gamma(1) = 1 \quad \Rightarrow \quad \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]

\[ \Rightarrow \quad \Omega_1 = 2\pi, \quad \Omega_2 = 4\pi, \quad \Omega_3 = 2\pi^2, \quad \text{etc.} \]
\[
I(a) = i \frac{2\pi^2}{16\pi^2} \int_0^\infty dk k^3 \frac{1}{[k^2 + a - ik]^2} \\
= \frac{i}{16\pi^2} \int_0^\infty dk k^2 \frac{1}{[k^2 + a - ik]^2}
\]

Need to cut off log divergence.

Rotationally invariant (i.e., Lorentz invariant) unless \(0 \leq k^2 \leq \Lambda^2\).

Now, integral is elementary,

\[
\int_0^\infty \frac{z}{(z+b)^2} \, dz = \int_0^\infty \frac{1}{z(b)} \, dz
\]

\[
= \left[ \ln(z) + \frac{b}{z} \right]_0^\infty
\]

\[
= \ln\left(\frac{\Lambda^2 + b}{b}\right) + \frac{b}{\Lambda^2 + b} - 1
\]

\[
= \ln\left(\frac{\Lambda^2}{b}\right) - 1 + O\left(\frac{b}{\Lambda^2}\right)
\]

So...

\[
I(a) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + a + ik]^2}
\]

\[
= \frac{i}{16\pi^2} \left[ \ln\left(\frac{\Lambda^2}{a + ik}\right) - 1 \right]
\]

(ignoring terms \(k^2 \to 0\)

\(\Lambda \to \infty\))

And so...

\[
\varphi = \frac{i\Lambda^2}{32\pi^2} \int_0^1 dx \left[ \ln\left(\frac{\Lambda^2}{(m^2 - x(1-x)p^2 - ik)^2}\right) - 1 \right]
\]
Here...

\[ S = (p_1 + p_2)^2 \]
\[ t = (p_1 + p_3)^2 \]
\[ u = (p_1 + p_4)^2 \]

So \( p^2 = S \). And...

\[ \lambda \chi + \chi + \chi \]

\[ = \frac{iA^2}{2\pi^2} \int_0^1 dx \left[ \ln \left( \frac{A^2}{m^2 - x(1-x)S - i\epsilon} \right) - 1 \right. \]
\[ \left. + (s \to t) + (s \to u) \right] \]

We fix the counterterm

\[ \lambda = -i \delta A \] by imposing suitable renormalization condition

Note

\[ p_1 + p_2 + p_3 + p_4 = 0 \Rightarrow 2 \sum_{i,j} p_i p_j = - \frac{1}{\epsilon} \]

\[ Z(s + t + u) = (p_1 + p_4)^2 + (p_3 + p_4)^2 - (p_1 + p_3)^2 - (p_2 + p_4)^2 \]
\[ + (p_1 + p_3)^2 + (p_2 + p_3)^2 \]

\[ = 2 \sum_{i,j} p_i p_j + 3 \sum p_i^2 \Rightarrow \sqrt{s + t + u} = \sqrt{p_1^2 + p_2^2 + p_3^2 + p_4^2} \]

Natural "symmetric point" at which to subtract

\[ s = t = u = \frac{4}{3} m^2 \]

i.e.

\[ \frac{\partial}{\partial \lambda} \mid_{s=t=u=\frac{4}{3} m^2} = -i \lambda \]

- Defines renormalized coupling
\[ \delta A = -3 \left( \sum_{n=1}^{\infty} \right) \frac{1}{r^2} = -3 \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ \delta A = \frac{3 \Lambda^2}{32 \pi^2} \int_0^1 dx \left[ \ln \left( \frac{\Lambda^2}{m^2 \left( 1 - \frac{1}{3} x (1-x) \right)} \right) - 1 \right] \]

\[ = \frac{3 \Lambda^2}{32 \pi^2} \left( \ln \frac{\Lambda^2}{m^2} + \text{finite} \right) + \text{higher order} \]

Pure number: \[ -\int_0^1 dx \ln \left( 1 - \frac{1}{3} x (1-x) \right) = -1 \]

After renormalization:

\[ -i \delta A = -i \frac{1}{32 \pi^2} \int_0^1 dx \left[ \ln \left( \frac{m^2 - x (1-x) s - i \epsilon}{m^2 \left( 1 - \frac{1}{3} x (1-x) \right)} \right) \right] \]

\[ + (s \to t) + (s \to u) \]

--- a finite expression ---

Note: In massless theory (m^2=0), subtracting at the symmetric point does not work. Need to choose an arbitrary subtraction mass -- e.g.

\[ \left( \sum \right) = -i \Lambda \]

s = t = u = m^2

(choose suitable momenta)
\[ \delta \chi = \frac{31^2}{32\pi^2} \int_0^1 dx \left[ \ln \frac{A^2}{\mu^2} - \ln x(1-x) - 1 \right] \]

\[ = \frac{31^2}{32\pi^2} \left( \ln \frac{A^2}{\mu^2} + 1 \right) \]

\[ \left( \int_0^1 dx \ln x = (x \ln x - x) \big|_0^1 = -1 \right) \]

\[ \text{and} \]

\[ \frac{-i\lambda^2}{32\pi^2} \left[ \ln \left( \frac{-s-i\epsilon}{\mu^2} \right) + \ln \left( \frac{-t-i\epsilon}{\mu^2} \right) + \ln \left( \frac{-u-i\epsilon}{\mu^2} \right) \right] \]

(quantum mechanics spoils classical scale invariance)

The renormalized coupling implicitly depends on \( \mu \) — and we can choose \( \mu \) to be whatever we like.

Goldstone & Low's key idea:
- Since \( \mu \) is arbitrary, we can choose it in any convenient way.

- And, indeed, we are free to subtract at an arbitrary point, even in the massive theory.

**Note:** expand around \( \phi^2 \sim \mu^2 \) in field renormalization:

\[ Z = Z \left( \frac{\Lambda^2}{\mu^2} \right) \]
Write the Lagrangian two ways

\[ L = \frac{1}{2} (\partial_\mu \phi) \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{i}{4!} \lambda \phi^4 \]

- in terms of bare mass, coupling, field

\[ \phi_0 = \sqrt{Z} \phi \]

\[ = \frac{1}{2} Z \partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{1}{2} m_0^2 Z \phi_0^2 - \frac{i}{4!} \lambda_0 Z \phi_0^4 \]

\[ = \frac{1}{2} (\partial_\mu \Phi) \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{i}{4!} \lambda \Phi^4 + L_{c.t.} \]

- in terms of renormal mass, coupling, field

\[ L_{c.t.} = (Z^{-1} \partial_\mu \Phi) \partial^\mu \Phi - \frac{1}{2} g_m^2 \Phi^2 - \frac{i}{4!} \delta \Phi^4 \]

So

\[ Z m_0^2 = m^2 + \delta m^2 \]

\[ Z^{-1} \lambda_0 = 1 + \delta \lambda \]

This is a different convention than on page 10 - more natural if we don't subtract on shell

\[ c.t. Feynman rule: \quad i \left( Z^{-1} \right)^2 p^2 - i \delta m^2 \times - i \delta \lambda \]

Renormalization is the procedure of writing

\[ L_{bare} = L_{renorm} + L_{counterterms} \]

This defines the theory

This division depends on conventions

Renormalization prescription is convenient
Renormalization is a convention-dependent reparameterization of the theory, performed (in part) so that calculated quantities make no reference to the cutoff.

Eq. — we used \[ \frac{1}{\ln(\frac{\Lambda}{\mu})} = -i \delta \]

\[ s = \epsilon + \mu^2 \]

Accepted

\[ \delta \mu = \frac{3}{32\pi^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) + 1 \]

(in number theory) + higher order

We also have

\[ Z = 1 + O(\delta \mu) \]

So

\[ \Lambda_0 = 1 + \frac{3}{32\pi^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) + O(\delta \mu^2) \]

\[ \delta \mu \]

\[ \text{convention-independent}
\]

bare parameter

\[ \text{renormalized parameter}
\]

\[ \text{must depend implicitly on } \mu
\]

E.g.

\[ \delta \mu + \frac{3}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2} = \delta \mu_0 + \frac{3}{32\pi^2} \ln \frac{\Lambda^2}{\mu_0^2} \]
The relation between the two renormalized couplings $\lambda$ and $\lambda'$ must not involve $A$.

Consider iteratively (implicitly regarding $\lambda' \ln (\Lambda'^2)$ as a small quantity)

$$
\lambda' = \lambda + \frac{3\lambda^2}{32\pi^2} (\ln \lambda^2 - \ln \lambda'^2)
$$

Substitute lowest order

$$
\lambda' = \lambda + \frac{3\lambda^2}{32\pi^2} \ln \frac{\lambda^2}{\lambda'^2} + \text{higher order}
$$

**Important:** $A$ does drop out (dependence is higher order). So higher order C.T. must cancel first remaining $A$-dependence.

$$
\beta(\lambda) = \lambda' \frac{d}{d\lambda} \lambda' \bigg|_{\lambda = \lambda'} = \frac{3\lambda^2}{16\pi^2} + O(1^3)
$$

**More traditional derivation:**

$$
\lambda \frac{d}{d\lambda} \lambda = 0 = \lambda \frac{d}{d\lambda} \left( \lambda + \frac{3\lambda^2}{32\pi^2} \ln \frac{\lambda^2}{\lambda'^2} + \text{h.o.} \right)
$$

$$
= \beta(\lambda) \left[ 1 + \frac{3\lambda^2}{16\pi^2} \ln \frac{\lambda^2}{\lambda'^2} \right] - \frac{3\lambda^2}{16\pi^2} + \text{h.o.}
$$

$$
\Rightarrow \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(1^3)
$$
Now -- let's integrate $\mu \frac{d\mu}{dt} = \beta(t)$

Consider

$\beta(t) = 6t^{n+1} + \ldots$

$\mu \frac{d\mu}{dt} = 6t^{n+1} + \ldots \Rightarrow 6 \ln \mu = \int \frac{d\mu}{t^{n+1}}$

$= \frac{1}{n \ln t} \ln \mu = \frac{1}{n} \left[ \ln \mu - \frac{1}{2} \ln^2 \mu \right]$

$\frac{1}{\mu} \ln \mu = \frac{1}{\mu} - \frac{1}{n} \ln \frac{\mu}{\lambda}

\Rightarrow \frac{1}{\mu} = \frac{1}{\mu} - \frac{1}{n} \ln \frac{\mu}{\lambda}$

Agrees with what we started with, when expanded, up to order $1^2$. Do higher order terms have any significance?

Yes! Summation of Taylor series

Suppose $\beta(t) = 60 t^2 + 6 t^3 + \ldots$

Solve $\frac{d\mu}{dt} = \beta(t)$ where $t = \ln \frac{\mu}{\lambda}$

By double power series expansion
\[ (\lambda/\theta) = \lambda_0 \]

\[ + \lambda_0^2 \left( C_{2,1} t \right) \]

\[ + \lambda_0^3 \left( C_{3,2} t^2 + C_{3,1} t \right) \]

(Note: \( \lambda_0 \) is not bare coupling -- it is \( \lambda \theta \)).

\[ \frac{d}{dt} \lambda = \lambda_0^2 (C_{2,1}) + \lambda_0^3 \left( 2C_{3,2} t + C_{3,1} \right) \]

\[ = 6_0 \left( \lambda_0 + \lambda_0^2 (C_{2,1} t) \right)^2 \]

\[ + 6_1 \left( \lambda_0 + \lambda_0^2 (C_{2,1} t) \right)^3 \]

Equate powers of \( \lambda \) to get \( C_{2,1} = 6_0 \)

\[ 6_0 \text{ determines } \int \lambda_0^3 t = 2C_{3,2} = 2 \times 6_0 \quad \Rightarrow \quad C_{3,2} = 6_0^2 \]

\[ 6_0 \text{ determines } \int \lambda_0^4 t = 3C_{3,1} = 6_0 (2C_{2,1} + 6_0 C_{3,2}) \]

\[ \Rightarrow \quad C_{3,1} = 6_0 \]

\[ 6_1 \text{ determines next-to-leading logs } \int \lambda_0^3 t = C_{3,1} = 6_1 \]

\[ \int \lambda_0^4 t = 2C_{4,2} = \frac{1}{6} \left( 6_0 (2C_{3,1}) + 6_1 3(C_{2,1}) \right) \]

\[ = 2 \times 6_0 6_1 + 3 \times 6_0 6_1 \]

\[ \Rightarrow \quad C_{4,2} = \frac{5}{2} \times 6_0 6_1 \]

---

Our formula for \( \lambda/\theta \) sums up the leading logs in each order in \( \lambda \).