Ph 219c/CS 219c

Exercises
Due: Wednesday 9 March 2011

3.1 What probability distributions are consistent with a mixed state?

A density operator ρ, expressed in the orthonormal basis $\{|\alpha_i\rangle\}$ that diagonalizes it, is

$$\rho = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \ . \tag{1}$$

We would like to realize this density operator as an ensemble of pure states $\{|\varphi_\mu\rangle\}$, where $|\varphi_\mu\rangle$ is prepared with a specified probability q_μ. This preparation is possible if the $|\varphi_\mu\rangle$'s can be chosen so that

$$\rho = \sum_\mu q_\mu |\varphi_\mu\rangle\langle\varphi_\mu| \ . \tag{2}$$

We say that a probability vector q (a vector whose components are nonnegative real numbers that sum to 1) is *majorized* by a probability vector p (denoted $q \prec p$), if there exists a *doubly stochastic* matrix D such that

$$q_\mu = \sum_i D_{\mu i} \ p_i \ . \tag{3}$$

A matrix is doubly stochastic if its entries are nonnegative real numbers such that $\sum_\mu D_{\mu i} = \sum_i D_{\mu i} = 1$. That the columns sum to one assures that D maps probability vectors to probability vectors (*i.e.*, is *stochastic*). That the rows sum to one assures that D maps the uniform distribution to itself. Applied repeatedly, D takes any input distribution closer and closer to the uniform distribution (unless D is a permutation, with one nonzero entry in each row and column). Thus we can view majorization as a partial order on probability vectors such that $q \prec p$ means that q is more nearly uniform than p (or equally close to uniform, in the case where D is a permutation).

Show that normalized pure states $\{|\varphi_\mu\rangle\}$ exist such that eq. (2) is satisfied if and only if $q \prec p$, where p is the vector of eigenvalues of ρ.

(Because the Shannon entropy is Schur concave, it follows that $H(q) \geq H(p) = H(\rho)$; that is, for any realization of the density operator ρ as an ensemble of pure states $\{|\varphi_\mu\rangle, q_\mu\}$, the Shannon entropy of the probability vector q is at least as large as the Von Neumann entropy of the density operator ρ.)

Hint: Recall that, according to the *Hughston-Jozsa-Wootters Theorem*, if eq. (1) and eq. (2) are both satisfied then there is a unitary matrix $V_{\mu i}$ such that

$$\sqrt{q_\mu}\ |\varphi_\mu\rangle = \sum_i \sqrt{p_i}\ V_{\mu i}|\alpha_i\rangle \tag{4}$$

(see Sec. 2.5.5 of the lecture notes). You may also use (but need not prove) *Horn's Lemma*: if $q \prec p$, then there exists a unitary (in fact, orthogonal) matrix $U_{\mu i}$ such that $q = Dp$ and $D_{\mu i} = |U_{\mu i}|^2$.

3.2 Positivity of quantum relative entropy

a) Show that $\ln x \leq x - 1$ for all positive real x, with equality iff $x = 1$.

b) The (classical) relative entropy of a probability distribution $\{p(x)\}$ relative to $\{q(x)\}$ is defined as

$$H(p \parallel q) \equiv \sum_x p(x) \left(\log p(x) - \log q(x)\right)\ . \tag{5}$$

Show that

$$H(p \parallel q) \geq 0\ , \tag{6}$$

with equality iff the probability distributions are identical. **Hint**: Apply the inequality from (*a*) to $\ln\left(q(x)/p(x)\right)$.

c) The quantum relative entropy of the density operator ρ with respect to σ is defined as

$$H(\rho \parallel \sigma) = \mathrm{tr}\ \rho \left(\log \rho - \log \sigma\right)\ . \tag{7}$$

Let $\{p_i\}$ denote the eigenvalues of ρ and $\{q_a\}$ denote the eigenvalues of σ. Show that

$$H(\rho \parallel \sigma) = \sum_i p_i \left(\log p_i - \sum_a D_{ia} \log q_a\right)\ , \tag{8}$$

where D_{ia} is a doubly stochastic matrix. Express D_{ia} in terms of the eigenstates of ρ and σ. (A matrix is doubly stochastic if its entries are nonnegative real numbers, where each row and each column sums to one.)

d) Show that if D_{ia} is doubly stochastic, then (for each i)

$$\log\left(\sum_a D_{ia} q_a\right) \geq \sum_a D_{ia} \log q_a \ , \tag{9}$$

with equality only if $D_{ia} = 1$ for some a.

e) Show that

$$H(\rho \,\|\, \sigma) \geq H(p \,\|\, r) \ , \tag{10}$$

where $r_i = \sum_a D_{ia} q_a$.

f) Show that $H(\rho \,\|\, \sigma) \geq 0$, with equality iff $\rho = \sigma$.

3.3 Properties of Von Neumann entropy

a) Use nonnegativity of quantum relative entropy to prove the *subadditivity* of Von Neumann entropy

$$H(\rho_{AB}) \leq H(\rho_A) + H(\rho_B), \tag{11}$$

with equality iff $\rho_{AB} = \rho_A \otimes \rho_B$. **Hint**: Consider the relative entropy of ρ_{AB} and $\rho_A \otimes \rho_B$.

b) Use subadditivity to prove the concavity of the Von Neumann entropy:

$$H(\sum_x p_x \rho_x) \geq \sum_x p_x H(\rho_x) \ . \tag{12}$$

Hint: Consider

$$\rho_{AB} = \sum_x p_x \left(\rho_x\right)_A \otimes \left(|x\rangle\langle x|\right)_B \ , \tag{13}$$

where the states $\{|x\rangle_B\}$ are mutually orthogonal.

c) Use the condition

$$H(\rho_{AB}) = H(\rho_A) + H(\rho_B) \quad \text{iff} \quad \rho_{AB} = \rho_A \otimes \rho_B \tag{14}$$

to show that, if all p_x's are nonzero,

$$H\left(\sum_x p_x \rho_x\right) = \sum_x p_x H(\rho_x) \tag{15}$$

iff all the ρ_x's are identical.

d) Use subadditivity to prove the triangle inequality:

$$H(\rho_{AB}) \geq |H(\rho_A) - H(\rho_B)| \ . \tag{16}$$

Hint: Construct a "purification" of ρ_{AB} — introduce a third system C and consider $|\Phi\rangle_{ABC}$ such that

$$\mathrm{tr}_C\left(|\Phi\rangle\langle\Phi|\right) = \rho_{AB} \ ; \tag{17}$$

then use the subadditivity relations $H(\rho_{BC}) \leq H(\rho_B) + H(\rho_C)$ and $H(\rho_{AC}) \leq H(\rho_A) + H(\rho_C)$.

3.4 Entanglement of typical bipartite pure states

Suppose that a pure state is chosen at random on the bipartite system AB, where $d_A/d_B \ll 1$. Then with high probability the density operator on A will be very nearly maximally mixed. The purpose of this problem is to derive this property.

To begin with, we will calculate the value of $\langle \mathrm{tr}\ \rho_A^2\rangle$, where $\langle\cdot\rangle$ denotes the average over all pure states $\{|\varphi\rangle\}$ of AB, and $\rho_A = \mathrm{tr}_B\left(|\varphi\rangle\langle\varphi|\right)$.

a) It is convenient to evaluate $\mathrm{tr}\ \rho_A^2$ using a trick. Imagine introducing a copy $A'B'$ of the system AB. Show that

$$\mathrm{tr}_A\ \rho_A^2 = \mathrm{tr}_{ABA'B'}\left[\left(S_{AA'} \otimes I_{BB'}\right)\left(|\varphi\rangle\langle\varphi|\right)_{AB} \otimes |\varphi\rangle\langle\varphi|_{A'B'})\right] \ , \tag{18}$$

where $S_{AA'}$ denotes the swap operator

$$S_{AA'} : |\varphi\rangle_A \otimes |\psi\rangle_{A'} \mapsto |\psi\rangle_A \otimes |\varphi\rangle_{A'} \ . \tag{19}$$

b) We wish to average the expression found in (*a*) over all pure states $|\varphi\rangle$. Rather than go into the details of how such an average is defined, I will simply assert that

$$\langle |\varphi\rangle\langle\varphi|_A \otimes |\varphi\rangle\langle\varphi|_{A'}\rangle = C\ \Pi_{AA'} \ , \tag{20}$$

where C is a constant and $\Pi_{AA'}$ denotes the projector onto the subspace of AA' that is *symmetric* under interchange of A and A'. Eq. (20) can be proved using invariance properties of the average and some group representation theory, but I hope you will regard it as obvious. The state being averaged is symmetric, and the average should not distinguish any symmetric state from any other symmetric state. Express the constant C in terms of the dimension $d \equiv d_A = d_{A'}$.

c) Use the property $\Pi_{AA'} = \frac{1}{2}(I_{AA'} + S_{AA'})$ to evaluate the expression found in (*a*). Show that

$$\left\langle \text{tr } \rho_A^2 \right\rangle = \frac{d_A + d_B}{d_A d_B + 1} \ . \tag{21}$$

d) Now estimate the average L^2 distance of ρ_A from the maximally mixed density operator $\frac{1}{d_A} I_A$, where $\| \ M \ \|_2 = \sqrt{\text{tr} M^\dagger M}$; show that

$$\left\langle \| \ \rho_A - \frac{1}{d_A} I_A \ \|_2 \right\rangle \leq \frac{1}{\sqrt{d_B}} \ . \tag{22}$$

Hints: First estimate $\left\langle \|\rho_A - \frac{1}{d_A} I_A\|_2^2 \right\rangle$ using eq. (21) and the obvious property $\langle \rho_A \rangle = \frac{1}{d_A} I_A$. Then show that for any nonnegative function f, it follows from the Cauchy-Schwarz inequality that $\langle \sqrt{f} \rangle \leq \sqrt{\langle f \rangle}$, and use this property to estimate $\left\langle \| \ \rho_A - \frac{1}{d_A} I_A \ \|_2 \right\rangle$.

e) Finally, estimate the average L^1 distance of ρ_A from the maximally mixed density operator, where $\| \ M \ \|_1 = \text{tr } \sqrt{M^\dagger M}$. Use the Cauchy-Schwarz inequality to show that $\| \ M \ \|_1 \leq \sqrt{d} \ \| \ M \ \|_2$, if M is a $d \times d$ matrix, and that therefore

$$\left\langle \| \ \rho_A - \frac{1}{d_A} I_A \ \|_1 \right\rangle \leq \sqrt{\frac{d_A}{d_B}} \ . \tag{23}$$

It follows from (*d*) that the average entanglement entropy of A and B is close to maximal for $d_A/d_B \ll 1$: $\langle H(A) \rangle \geq \log_2 d_A - d_A/2d_B \ln 2$, though you are not asked to prove this bound. Thus, if for example A is 50 qubits and B is 100 qubits, the typical entropy deviates from maximal by only about $2^{-50} \approx 10^{-15}$.