Ph219C/CS219C

Exercises
Due: Thursday 5 May 2022

2.1 A quantum version of Fano's inequality

a) In a d-dimensional system, suppose a density operator $\boldsymbol{\rho}$ approximates the pure state $|\psi\rangle$ with fidelity

$$F = \langle\psi|\boldsymbol{\rho}|\psi\rangle = 1 - \varepsilon. \tag{1}$$

Show that

$$H(\boldsymbol{\rho}) \leq H_2(\varepsilon) + \varepsilon \log_2(d-1). \tag{2}$$

Hint: Recall that if a complete orthogonal measurement performed on the state $\boldsymbol{\rho}$ has distribution of outcomes X, then $H(\boldsymbol{\rho}) \leq H(X)$, where $H(X)$ is the Shannon entropy of X.

b) As in §10.7.2, suppose that the noisy channel $\mathcal{N}^{A\to B}$ acts on the pure state ψ_{RA}, and is followed by the decoding map $\mathcal{D}^{B\to C}$. Show that

$$H(R)_{\boldsymbol{\rho}} - I_c(R\,\rangle B)_{\boldsymbol{\rho}} \leq 2H(RC)_{\boldsymbol{\sigma}}, \tag{3}$$

where

$$\boldsymbol{\rho}_{RB} = \mathcal{N}(\psi_{RA}), \quad \boldsymbol{\sigma}_{RC} = \mathcal{D}\circ\mathcal{N}(\psi_{RA}). \tag{4}$$

Therefore, if the decoder's output (the state of RC) is almost pure, then the coherent information of the channel $\mathcal{N}$ comes close to matching its input entropy. **Hint**: Use the data processing inequality $I_c(R\,\rangle C)_{\boldsymbol{\sigma}} \leq I_c(R\,\rangle B)_{\boldsymbol{\rho}}$ and the subadditivity of von Neumann entropy. It is convenient to consider the joint pure state of the reference system, the output, and environments of the dilations of $\mathcal{N}$ and $\mathcal{D}$.

c) Suppose that the decoding map recovers the channel input with high fidelity,

$$F(\mathcal{D}\circ\mathcal{N}(\psi_{RA}), \psi_{RC}) = 1 - \varepsilon. \tag{5}$$

Show that

$$H(R)_{\boldsymbol{\rho}} - I_c(R\,\rangle B)_{\boldsymbol{\rho}} \leq 2H_2(\varepsilon) + 2\varepsilon \log_2(d^2-1), \tag{6}$$

assuming that R and C are d-dimensional. This is a quantum version of Fano's inequality, which we may use to derive an upper bound on the quantum channel capacity of $\mathcal{N}$.

2.2 Noisy superdense coding and teleportation.

a) By converting the entanglement achieved by the mother protocol into classical communication, prove the noisy superdense coding resource inequality:

$$Noisy\ SD: \quad \langle \phi_{ABE} \rangle + H(A)[q \to q] \geq I(A;B)[c \to c]. \tag{7}$$

Verify that this matches the standard noiseless superdense coding resource inequality when ϕ is a maximally entangled state of AB.

b) By converting the entanglement achieved by the mother protocol into quantum communication, prove the noisy teleportation resource inequality:

$$Noisy\ TP: \quad \langle \phi_{ABE} \rangle + I(A;B)[c \to c] \geq I_c(A\rangle B)[q \to q]. \tag{8}$$

Verify that this matches the standard noiseless teleportation resource inequality when ϕ is a maximally entangled state of AB.

2.3 Degradability of amplitude damping and erasure

The qubit amplitude damping channel $\mathcal{N}_{\text{a.d.}}^{A\to B}(p)$ discussed in §3.4.3 has the dilation $\boldsymbol{U}^{A\to BE}$ such that

$$\begin{aligned} \boldsymbol{U}: &|0\rangle_A \mapsto |0\rangle_B \otimes |0\rangle_E, \\ &|1\rangle_A \mapsto \sqrt{1-p}\;|1\rangle_B \otimes |0\rangle_E + \sqrt{p}\;|0\rangle_B \otimes |1\rangle_E; \end{aligned}$$

a qubit in its "ground state" $|0\rangle_A$ is unaffected by the channel, while a qubit in the "excited state" $|1\rangle_A$ decays to the ground state with probability p, and the decay process excites the environment. Note that $\boldsymbol{U}$ is invariant under interchange of systems B and E accompanied by transformation $p \leftrightarrow (1-p)$. Thus the channel complementary to $\mathcal{N}_{\text{a.d.}}^{A\to B}(p)$ is $\mathcal{N}_{\text{a.d.}}^{A\to E}(1-p)$.

a) Show that $\mathcal{N}_{\text{a.d.}}^{A\to B}(p)$ is degradable for $p \le 1/2$. Therefore, the quantum capacity of the amplitude damping channel is its optimized one-shot coherent information. **Hint**: It suffices to show that

$$\mathcal{N}_{\text{a.d.}}^{A\to E}(1-p) = \mathcal{N}_{\text{a.d.}}^{B\to E}(q) \circ \mathcal{N}_{\text{a.d.}}^{A\to B}(p), \tag{9}$$

where $0 \le q \le 1$.

The *erasure channel* $\mathcal{N}_{\text{erase}}^{A\to B}(p)$ has the dilation $\boldsymbol{U}^{A\to BE}$ such that

$$\boldsymbol{U} : |\psi\rangle_A \mapsto \sqrt{1-p}\, |\psi\rangle_B \otimes |e\rangle_E + \sqrt{p}\, |e\rangle_B \otimes |\psi\rangle_E; \tag{10}$$

Alice's system passes either to Bob (with probability $1-p$) or to Eve (with probability p), while the other party receives the "erasure symbol" $|e\rangle$, which is orthogonal to Alice's Hilbert space. Because $\boldsymbol{U}$ is invariant under interchange of systems B and E accompanied by transformation $p \leftrightarrow (1-p)$, the channel complementary to $\mathcal{N}_{\text{erase}}^{A\to B}(p)$ is $\mathcal{N}_{\text{erase}}^{A\to E}(1-p)$.

b) Show that $\mathcal{N}_{\text{erase}}^{A\to B}(p)$ is degradable for $p \le 1/2$. Therefore, the quantum capacity of the erasure channel is its optimized one-shot coherent information. **Hint**: It suffices to show that

$$\mathcal{N}_{\text{erase}}^{A\to E}(1-p) = \mathcal{N}_{\text{erase}}^{B\to E}(q) \circ \mathcal{N}_{\text{erase}}^{A\to B}(p), \tag{11}$$

where $0 \le q \le 1$.

c) Show that for $p \le 1/2$ the quantum capacity of the erasure channel is

$$Q(\mathcal{N}_{\text{erase}}^{A\to B}(p)) = (1-2p)\log_2 d, \tag{12}$$

where A is d-dimensional, and that the capacity vanishes for $1/2 \le p \le 1$.

2.4 Proof of the decoupling inequality

In this problem we complete the derivation of the decoupling inequality sketched in §10.9.1. Equation numbers of the form (10.xxx) refer to Chapter 10 of the lecture notes.

a) Verify eq.(10.336).

To derive the expression for $\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right]$ in eq.(10.340), we first note that the invariance property eq.(10.325) implies that $\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right]$ commutes with $\boldsymbol{V}\otimes\boldsymbol{V}$ for any unitary $\boldsymbol{V}$. Therefore, by Schur's lemma, $\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right]$ is a weighted sum of projections onto irreducible representations of the unitary group. The tensor product of two fundamental representations of $\boldsymbol{U}(d)$ contains two irreducible representations — the symmetric and antisymmetric tensor representations. Therefore we may write

$$\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right] = c_{\mathrm{sym}}\ \boldsymbol{\Pi}^{(\mathrm{sym})}_{AA'} + c_{\mathrm{anti}}\ \boldsymbol{\Pi}^{(\mathrm{anti})}_{AA'}; \tag{13}$$

here $\boldsymbol{\Pi}^{(\mathrm{sym})}_{AA'}$ is the orthogonal projector onto the subspace of AA' symmetric under the interchange of A and A', $\boldsymbol{\Pi}^{(\mathrm{anti})}_{AA'}$ is the projector onto the antisymmetric subspace, and c_{sym}, c_{anti} are suitable constants. Note that

$$\begin{aligned}
\boldsymbol{\Pi}^{(\mathrm{sym})}_{AA'} &= \frac{1}{2}\left(\boldsymbol{I}_{AA'} + \boldsymbol{S}_{AA'}\right), \\
\boldsymbol{\Pi}^{(\mathrm{anti})}_{AA'} &= \frac{1}{2}\left(\boldsymbol{I}_{AA'} - \boldsymbol{S}_{AA'}\right),
\end{aligned} \tag{14}$$

where $\boldsymbol{S}_{AA'}$ is the swap operator, and that the symmetric and antisymmetric subspaces have dimension $\frac{1}{2}|A|\left(|A|+1\right)$ and dimension $\frac{1}{2}|A|\left(|A|-1\right)$ respectively.

Even if you are not familiar with group representation theory, you might regard eq.(13) as obvious. We may write $\boldsymbol{M}_{AA'}(\boldsymbol{U})$ as a sum of two terms, one symmetric and the other antisymmetric under the interchange of A and A'. The expectation of the symmetric part must be symmetric, and the expectation value of the antisymmetric part must be antisymmetric. Furthermore, averaging over the unitary group ensures that no symmetric state is preferred over any other.

b) To evaluate the constant c_{sym}, multiply both sides of eq.(13) by $\boldsymbol{\Pi}^{(\mathrm{sym})}_{AA'}$ and take the trace of both sides, thus finding

$$c_{\mathrm{sym}} = \frac{|A_1| + |A_2|}{|A|+1}. \tag{15}$$

c) To evaluate the constant c_{anti}, multiply both sides of eq.(13)) by $\boldsymbol{\Pi}^{(\mathrm{anti})}_{AA'}$ and take the trace of both sides, thus finding

$$c_{\mathrm{anti}} = \frac{|A_1| - |A_2|}{|A|-1}. \tag{16}$$

d) Using

$$c_{\boldsymbol{I}} = \frac{1}{2}\left(c_{\rm sym} + c_{\rm anti}\right), \quad c_{\boldsymbol{S}} = \frac{1}{2}\left(c_{\rm sym} - c_{\rm anti}\right) \tag{17}$$

prove eq.(10.341).