Ph219C/CS219C

Exercises
Due: Thursday 18 April 2024

1.1 Positivity of quantum relative entropy

a) Show that $\ln x \le x-1$ for all positive real x, with equality iff $x = 1$.

b) The (classical) relative entropy of a probability distribution $\{p(x)\}$ relative to $\{q(x)\}$ is defined as

$$D(p\|q) \equiv \sum_x p(x)\left(\log p(x) - \log q(x)\right) \ . \tag{1}$$

Show that

$$D(p\|q) \ge 0 \ , \tag{2}$$

with equality iff the probability distributions are identical. **Hint**: Apply the inequality from (a) to $\ln\left(q(x)/p(x)\right)$.

c) The quantum relative entropy of the density operator $\boldsymbol{\rho}$ with respect to $\boldsymbol{\sigma}$ is defined as

$$D(\boldsymbol{\rho}\|\boldsymbol{\sigma}) = \mathrm{tr}\ \boldsymbol{\rho}\left(\log\boldsymbol{\rho} - \log\boldsymbol{\sigma}\right) \ . \tag{3}$$

Let $\{p_i\}$ denote the eigenvalues of $\boldsymbol{\rho}$ and $\{q_a\}$ denote the eigenvalues of $\boldsymbol{\sigma}$. Show that

$$D(\boldsymbol{\rho}\|\boldsymbol{\sigma}) = \sum_i p_i\left(\log p_i - \sum_a D_{ia}\log q_a\right) \ , \tag{4}$$

where D_{ia} is a doubly stochastic matrix. Express D_{ia} in terms of the eigenstates of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. (A matrix is doubly stochastic if its entries are nonnegative real numbers, where each row and each column sums to one.)

d) Show that if D_{ia} is doubly stochastic, then (for each i)

$$\log\left(\sum_a D_{ia}q_a\right) \ge \sum_a D_{ia}\log q_a \ , \tag{5}$$

with equality only if $D_{ia} = 1$ for some a.

e) Show that

$$D(\boldsymbol{\rho}\|\boldsymbol{\sigma}) \geq D(p\|r)\ , \tag{6}$$

where $r_i = \sum_a D_{ia} q_a$.

f) Show that $D(\boldsymbol{\rho}\|\boldsymbol{\sigma}) \geq 0$, with equality iff $\boldsymbol{\rho} = \boldsymbol{\sigma}$.

1.2 Properties of Von Neumann entropy

a) Use nonnegativity of quantum relative entropy to prove the *subadditivity* of Von Neumann entropy

$$H(\boldsymbol{\rho}_{AB}) \leq H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_B), \tag{7}$$

with equality iff $\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$. **Hint**: Consider the relative entropy of $\boldsymbol{\rho}_{AB}$ and $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$.

b) Use subadditivity to prove the concavity of the Von Neumann entropy:

$$H(\sum_x p_x \boldsymbol{\rho}_x) \geq \sum_x p_x H(\boldsymbol{\rho}_x)\ . \tag{8}$$

Hint: Consider

$$\boldsymbol{\rho}_{AB} = \sum_x p_x \left(\boldsymbol{\rho}_x\right)_A \otimes \left(|x\rangle\langle x|\right)_B\ \ , \tag{9}$$

where the states $\{|x\rangle_B\}$ are mutually orthogonal.

c) Use the condition

$$H(\boldsymbol{\rho}_{AB}) = H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_B) \quad \text{iff} \quad \boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B \tag{10}$$

to show that, if all p_x's are nonzero,

$$H\left(\sum_x p_x \boldsymbol{\rho}_x\right) = \sum_x p_x H(\boldsymbol{\rho}_x) \tag{11}$$

iff all the $\boldsymbol{\rho}_x$'s are identical.

1.3 Monotonicity of quantum relative entropy

Quantum relative entropy has a property called *monotonicity*:

$$D(\boldsymbol{\rho}_A\|\boldsymbol{\sigma}_A) \leq D(\boldsymbol{\rho}_{AB}\|\boldsymbol{\sigma}_{AB}); \tag{12}$$

The relative entropy of two density operators on a system AB cannot be less than the induced relative entropy on the subsystem A.

a) Use monotonicity of quantum relative entropy to prove the strong subadditivity property of Von Neumann entropy. **Hint**: On a tripartite system ABC, consider the relative entropy of $\boldsymbol{\rho}_{ABC}$ and $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_{BC}$.

b) Use monotonicity of quantum relative entropy to show that the action of a quantum channel $\mathcal{N}$ cannot increase relative entropy:

$$D(\mathcal{N}(\boldsymbol{\rho})\|\mathcal{N}(\boldsymbol{\sigma}) \leq D(\boldsymbol{\rho}\|\boldsymbol{\sigma}), \tag{13}$$

Hint: Recall that any quantum channel has an isometric dilation.

1.4 Hypothesis testing

The (classical) relative entropy $D(p\|q)$ charactizes the difference between the distributions $p(x)$ and $q(x)$ in a sense that is useful for hypothesis testing. Suppose that an unknown distribution is sampled $n \ggg 1$ times, and we find that outcome x occurs $np(x)$ times. We would like to assess whether these outcomes are compatible with sampling from a hypothetical distribution $q(x)$. As discussed in class, the probability of obtaining this result if we are actually sampling from $q(x)$ may be estimated as

$$\text{Probability} = 2^{-nD(p\|q)}. \tag{14}$$

a) Suppose

$$q(x) = p(x) + \varepsilon(x), \quad \sum_x \varepsilon(x) = 0. \tag{15}$$

Write $D(p\|q)$ as a function of $p(x)$ and $\varepsilon(x)$, expanded to quadratic order in $\varepsilon(x)$. For this purpose, suppose that the logarithms in the definition of relative entropy are natural logs rather than logs to the base 2.

b) Suppose that

$$\|\varepsilon\|_1 = \sum_x |\varepsilon(x)| = \delta. \tag{16}$$

Using the quadratic approximation found in (*a*), find the minimal value of $D(p\|q)$ and the hypothetical distribution $q(x) = p(x) + \varepsilon(x)$ that minimizes it.

c) Suppose we toss a coin n times, and the outcome "heads" is observed $n\left(\frac{1}{2} + \frac{\delta}{2}\right)$ times. Hence $p(\text{heads}) = \frac{1}{2}+\frac{\delta}{2}$ and $p(\text{tails}) = \frac{1}{2}-\frac{\delta}{2}$. Consider the hypothesis that the coin is unbiased: $q(\text{heads}) = q(\text{tails}) = \frac{1}{2}$. Compute the relative entropy $D(p\|q)$ to quadratic order in δ.

d) A coin is tossed 1 million times. Use the result of (*c*) to estimate the probability of the coin coming up heads 505,000 times, assuming that the coin is unbiased.

1.5 The first law of Von Neumann entropy

We'll use $S(\boldsymbol{\rho}) = -\text{tr}\,(\boldsymbol{\rho} \ln \boldsymbol{\rho})$ to denote the entropy of a density operator when using natural logarithms instead of logarithms with base 2. As in §10.2.6, a $d \times d$ density matrix can be expressed as

$$\boldsymbol{\rho} = \frac{e^{-\boldsymbol{K}}}{\text{tr}\left(e^{-\boldsymbol{K}}\right)}, \tag{17}$$

where $\boldsymbol{K}$ is a $d \times d$ Hermitian matrix called the *modular Hamiltonian* associated with $\boldsymbol{\rho}$. (Under this definition of $\boldsymbol{K}$, we have the freedom to shift $\boldsymbol{K}$ by a multiple of the identity operator without changing $\boldsymbol{\rho}$.) We assume that $\boldsymbol{\rho}$ has full rank; that is, it has d positive eigenvalues. We will see that when $\boldsymbol{\rho}$ changes slightly, the first-order change in $S(\boldsymbol{\rho})$ can be related to the change in the expectation value of $\boldsymbol{K}$.

a) Suppose $\boldsymbol{A}(\lambda)$ is a bounded Hermitian operator smoothly parametrized by the real number λ. Show that

$$\frac{d}{d\lambda}\left(\text{tr}\boldsymbol{A}^n\right) = n\ \text{tr}\left(\frac{d\boldsymbol{A}}{d\lambda}\boldsymbol{A}^{n-1}\right). \tag{18}$$

Do not assume that $d\boldsymbol{A}/d\lambda$ commutes with $\boldsymbol{A}$.

b) Suppose the density operator is perturbed slightly:

$$\boldsymbol{\rho} \to \boldsymbol{\rho}' = \boldsymbol{\rho} + \delta\boldsymbol{\rho}. \tag{19}$$

Since $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are both normalized density operators, we have $\text{tr}\,(\delta\boldsymbol{\rho}) = 0$. Show that

$$S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}) = \text{tr}\left(\boldsymbol{\rho}'\boldsymbol{K}\right) - \text{tr}\left(\boldsymbol{\rho}\boldsymbol{K}\right) + O\left((\delta\boldsymbol{\rho})^2\right); \tag{20}$$

that is,

$$\delta S = \delta\langle\boldsymbol{K}\rangle \tag{21}$$

to first order in the small change in $\boldsymbol{\rho}$. This statement generalizes the first law of thermodynamics; for the case of a thermal density operator with $\boldsymbol{K} = \boldsymbol{H}/T$ (where $\boldsymbol{H}$ is the Hamiltonian and T is the temperature), it becomes the more familiar statement

$$\delta E = \delta\langle\boldsymbol{H}\rangle = T\delta S. \tag{22}$$