General Relativity

- How it relates to classical relativity
  Newton’s laws of motion take the same form for observers at rest and in uniform motion

- You can’t tell if you are moving, if no acceleration

- But does principle apply to all laws of physics?
  What don’t work well?

\[ c = \frac{1}{\sqrt{\mu c^2}} \]

- Einstein 1905 - universality
  There are inertial frames, in uniform motion relative to one another
  All non-gravitational laws of physics have a universal form, independent of choice of inertial frame

- C = I in all inertial frames
  (You can’t use light to tell if you are moving or not!)

A principle of universality – but came to be called “relativity” – because space and time are perceived differently by different observers.
But what about gravity?

Newton's theory \( F = G \frac{m_1 m_2}{r^2} \)
does not obey the principle — or
\[
F = \nabla \phi \quad \phi = \frac{m}{r} \quad \nabla^2 \phi = 4\pi \rho
\]

-\( \phi \) represents a scalar
Neumann potential density
-\( \rho \) is a constant
\( \phi \) a tensor \( T_{\mu\nu} \)

Turned out — to require
a much more involved approach than most suspected

\underline{Einstein 1907}

makes first serious attempt to formulate
relativistic theory of gravitation

\( \Box \phi = 0 \)

Conservation of energy and momentum in local form

1907: "The happiest thought of my life!"
someone falling freely does not feel his own weight

The remarkable hint is
Principle of Equivalence of Gravitational and Inertial
\( \text{Mass} = \text{Momentum} \)

Bodies of different masses feel the same way!!

\( T - \rho \) verified experimentally to
only within 10^-7 accuracy?
All freely falling observers perceive the same physical laws (in sufficiently small region of space and time).

You can't tell if you are in a rocket accelerating with \( a = \frac{g}{2} \) or a gravitational field \( g \).

So - what is gravity? An artifact of our moving the wrong way? No - there are observers independent of physical effects, due to local variations in \( g \) - or tidal forces. E.g., the moon causes ocean tides.

Powerful enough to tear apart a (freely falling) moon of Jupiter.

Principle of Equivalence

\[ \leftrightarrow \text{Principle of "general covariance" (see below)} \]

In locally inertial frames, laws of physics are same as in absence of gravitation.

(Becomes same as general rule of tensor calculus.)

What's the principle of equivalence? Not an invariance principle (cf Lorentz invariance). Rather - a gauge principle - tells us how gravity couples...
A relative acceleration, proportional to relative separation,
$$a = \dot{\delta} \times \delta$$

Gradually, AE recognized the analogy with Riemannian geometry.

- Relativity: Identify the invariant features of physics, that don't depend on state of motion of observer, or more generally, on how observer chooses coordinates.

- Geometry: What are the invariant ("intrinsic") properties of a space, not depending on how space is parametrized.

"Smooth Manifold" in a local patch, looks like a "flat" Euclidean space.

(And in Riemannian geometry, there are "measuring rods" that provide a local measure of distance between neighboring points:
$$ds^2 = S_{ij} d\xi^i d\xi^j \equiv d\xi^2 + d\xi^2$$)

But curvature is the intrinsic concept.
At each point, a "tangent space" we can carry a parallel transport a vector around a closed path. Now points in a different direction.

Every a swinging "pendulum on the earth's surface" winds up pointing a different way.

- This is intrinsic curvature. The could be measured by someone who never leaves the surface.

- As opposed to extrinsic curvature of cylinders - a property of how it is embedded.

\[ \text{Example: great circle on a sphere} \]

\[ \text{Remark: geodesics have extremal length.} \]

\[ \text{A geodesic is a path that is locally straight.} \]

\[ \text{Use an invariance rod to mark off increments of constant length \( k \).} \]

\[ \text{Consider 2 nearby geodesics and \( \vec{s}(l) \), connecting them.} \]

\[ \text{Flat space} \quad \vec{s}(l) = a + b \text{ or } \vec{s} = 0 \]

\[ \text{Curved space} \quad \vec{s} \neq 0 \]
\[ \dot{\delta} = (\text{curvature}) \delta \]

where curvature determines how geodesics attract (positive curvature)
or repel (negative curvature)

\[ \text{E.g.} \quad \theta_1 + \theta_2 + \theta_3 > \pi \]

\[ \theta_1 + \theta_2 + \theta_3 < \pi \]

Summary of Riemannian Geometry:
- Space is approximately flat locally, locally Euclidean.
- Can parallel transport a vector, but at a cost: when we transport around a closed path, on different paths gives different results.
- "Path dependence of parallel transport"
  - Geodesic deviation
  - Gravitational analogy:
    - Space is locally Minkowski
    - Locally Euclidean charts \( \leftrightarrow \) local inertial frame
      - Gyroscope = parallel transport
        - More time
        - Path dependent?
        - Free fall = geodesic
          - Free fall: \( \dot{s} = (\text{curvature}) s \)
          - Tidal forces
            - (cancels you a pull you apart)

\[ \text{E.g.} \quad \theta_1 + \theta_2 + \theta_3 > \pi \]

\[ \theta_1 + \theta_2 + \theta_3 < \pi \]
Tensors

In both RG and GR, the aim is to separate intrinsic (coordinate-independent) properties from special properties that hold only in particular coordinate systems.

In fact, coordinate-independent properties are expressed in terms of tensors.

In principle, laws of physics should be expressed in coordinate-independent way.

In practice, we need to use coordinates to solve the equations!

We can express a tensor in particular coordinates. Then, what makes it a tensor is how it transforms when we change the coordinate system.

A tensor is something that transforms like a tensor. (?)

With tensor technology, the principle of equivalence becomes:

"The principle of general covariance."
PAE: "To learn physics look for same to all freely falling observers."

P QGC:
An equation

1. Held in "flat space"
2. Is a tensor equation

⇒ It holds in curved space (including gravity)

E.Q. $\mathcal{T}^\mu = \text{Tensor}$

If $\mathcal{T}^\mu = 0$

Then $\mathcal{T}^\mu = 0$

And that is why we care about tensors in physics -- they enable us to promote laws from a particular coordinate system to laws that hold in general (any coordinate system).

But what is a tensor (i.e., how does it transform)?

$x^\mu = \text{old set of coordinates} \Rightarrow x'^\mu(x) = \text{new set}$

Consider an infinitesimal displacement

$\Delta x^\mu \rightarrow \Delta x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \Delta x^\nu$ "contravariant" index
Consider quadratic
\[
\frac{\partial}{\partial x^k} \rightarrow \frac{\partial}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \quad \text{"Covariant" index}
\]

A general tensor has
up (contravariant) and down (covariant) indices

\[
T(x) \alpha \beta \gamma \rightarrow \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial x^\beta} \frac{\partial x^\delta}{\partial x^\gamma}
\]

Evidently

\[
T(x) = 0 \Rightarrow T'(x') = 0 \quad \text{key property of a Tensor}
\]

But what is the coordinate-independent meaning of a Tensor

Two basic types of Tensors

- A vector (field)

\[ \vec{V} \]

Assigns a tangent vector to each point on the manifold

One way to think about a vector field

Consider parameterized flow in the space (like the flow parameterized by the Hamiltonian mechanics)
Then \( \frac{d}{dx} \) is a vector field. It assigns to each point a tangent vector:
- direction = direction of flow
- length = rate of flow

Thus it has a meaning independent of choice of coordinates (however, length of vector depends on how the flow is parameterized).

What if we choose particular coordinates? \( x^m(x) \) describes flow in vicinity of a point.

\[
\frac{d}{dt} \left( x^m(x) \right) = \frac{dx^m}{dt} = \frac{dx^m}{dx^i} \frac{dx^i}{dt} = \frac{dx^m}{dt} \cdot e_m
\]

This is expansion of vector field in terms of a particular basis. The basis \( e_m \) is the tangent space defined by the coordinates \( x^m \).

If we used different coordinates, then the same vector field could be expressed as

\[
\frac{dx^m}{dx^i} \frac{\partial}{\partial x^i} = \frac{dx^m}{dx^i} \frac{\partial}{\partial x^i} = \frac{\partial x^m}{\partial x^i} \frac{\partial}{\partial x^i}
\]

\[
\mathbf{V} = V^m e_m = V^i e_i
\]

Component of basis \( \mathbf{V} \) in \( \mathbf{V} \) is \( \frac{\partial x^m}{\partial x^i} V^m \) — component contra-variant.
The other basic type of tensor is a one-form.

It is a linear map that takes a vector to a real number:

\[ \langle \omega, \mathbf{v} \rangle = \text{number} \]

Example: \( \phi \) = scalar field (a number at each point)

\( d\phi \) is a 1-form

\[ \langle d\phi, \frac{d}{dt} \rangle = \frac{d\phi}{dt} \]

Note that:
Vectors and 1-forms are dual to each other.
1-form: vector \( \rightarrow \) number
Vector: 1-form \( \rightarrow \) number

We can expand 1-form in a basis determined by choice of coordinates:

\[ d\phi = \sum_{\mu} \frac{\partial \phi}{\partial x^\mu} \, dx^\mu \]
\[ \left< d x^m, \frac{\partial}{\partial x^m} \right> = \delta^m_n \]

Hence \( \left< d \phi, \frac{\partial}{\partial x^m} \right> = \left< d \phi, d x^m, \frac{\partial}{\partial x^m} \right> = \frac{d \phi}{d x^m} \)

coordinate-free view

of a tensor:

\[ T^i (\sigma, \sigma, 0, 0, \sigma, 0, \sigma, \sigma) = \text{Tensor} \]

\[ \text{vector} \rightarrow \text{one-form} \]

(Multi-)

Linear map: vectors + one forms to numbers

E.g., \( T^i (\sigma) = L \)-form

\[ \text{vector} \rightarrow \text{vector} \]

\[ T^i (\sigma) = \text{vector} \]

\[ \text{vector} \rightarrow \text{one-form} \]

contravariant (vector)

\[ \text{one-form} \rightarrow \text{one-form} \]

\[ \frac{1}{\sqrt{\text{det}(g)}} \epsilon_{\mu \nu \alpha \beta} e^\mu \epsilon e^\nu \epsilon d x^\alpha d x^\beta \]

\( T \) devours the vector + 1-form with胜利 by contraction of indices
\[
\hat{T} (\mathbf{V}, \omega) = T (V^x, \omega \, dx^\mu) = T^{\mu}_{\mu} V^\mu \quad \text{number}
\]

A tensor can also denote a tensor (or the same or lower rank).

Eq. \( T^{\mu \nu} \alpha \beta \quad \text{has 2 unfilled slots -- two a tensor}

An \((m)\) tensor has \(n\) up indices (1-forms) in down indices (slots for vectors).

\underline{Metric Tensor}

Is the fundamental tensor of Riemannian geometry -- it is a \((2)\) tensor \( g (\mathbf{U}, \mathbf{V}) = g_{\mu \nu} \, dx^\mu \, dx^\nu \)

Interpretation \( g \) as inner product of two tangent vectors.

In components \( g = g_{\mu \nu} \, dx^\mu \, dx^\nu \)
\[ g(\mathbf{u}, \mathbf{v}) = g_{\mu \nu} u^\mu v^\nu \]

There are locally Euclidean (or Minkowski) coordinates such that

\[ g = \eta_{\alpha \beta} \, dx^\alpha \otimes dx^\beta \]

\[ \eta(x) = \text{locally inertial coords} \]

Hence,

\[ g = \eta_{\alpha \beta} \frac{2 \eta_{\mu \nu} \, dx^\mu \otimes dx^\nu}{\partial x^\alpha / \partial x^\beta} \]

\[ g_{\mu \nu} = \text{in terms of locally flat coordinates} \]

The metric tensor (or any \((0, 2)\) tensor) defines an isomorphism relating 1-forms and vectors

\[ \mathbf{v} \leftrightarrow g(\mathbf{v}, \cdot) = \mathbf{w} \]

(i.e., \( \mathbf{w} = g(\mathbf{v}, \cdot) \))

In component language, we can "raise" and "lower" indices with \( g \)

\[ w^\alpha = g_{\alpha \beta} v^\beta \]

\[ \text{lowers an index vector} \rightarrow 1\text{-form} \]
We can also define a \((2,0)\) tensor such that
\[
V^\beta = g^{\beta \alpha} W_\alpha = g^{\beta \alpha} g_{\alpha \gamma} V^\gamma
\]
\[
\Rightarrow g^{\beta \alpha} g_{\alpha \gamma} = \delta^\beta_\gamma
\]

where \(g^{\beta \alpha}\) is the inverse matrix of \(g_{\alpha \beta}\),

\[
f = g^{\alpha \beta} e_\alpha e_\beta = f(\ast, \ast)\]

is a \((2,0)\) tensor such that
\[
V^\ast = f(W \ast, \ast)
\]

invertible map
\(\text{vector} \rightarrow \text{1-form}\)

So we can raise and lower indices of \(\ast\) tensor with \(g^{\alpha \beta}\).

By convention, if \(T^\alpha_\beta\) is a \((2,0)\) tensor then \(T^\alpha_\beta\) denotes the \((1,1)\) tensor
\[
T^\alpha_\beta = g^{\beta \gamma} T^\alpha_\gamma
\]
Covariant Derivative

We have already discussed differentiation of a scalar
\[ \frac{d \phi}{d \lambda} = \frac{d \phi}{d \lambda} \]

How do we differentiate vectors?

This is tricky -- because vectors at neighboring points lie in different tangent spaces.

To compose, we need a notion of parallel transport (a "connection") that relates these two vector spaces.

- A connection is a coordinate-independent concept that tells us how a vector at A becomes a certain vector upon carried to B.

- Given a connection, there is a coordinate-independent notion of a derivative of a vector field.

Compose vector at B to result of transporting vector at A to B -- this is how the vector changes.
along the flow from A to B.

The covariant derivative of a vector field is a (1,1) tensor, it has a slot for a vector — tells us how the vector field changes along flow determined by vector in the slot

\[ \nabla \mathbf{u} \]

Is a vector field that tells us how \( \mathbf{v} \) changes (relative to parallel transport) along flow \( \mathbf{u} \)

In Riemannian geometry, the notion of parallel transport is provided by the locally flat structure (by the metric tensor)

If \( \mathbf{v}, \mathbf{w} \) are parallel transported along \( \mathbf{u} \), then

\[ g(\mathbf{v}, \mathbf{w}) = \text{constant along } \mathbf{u} \]

Or -- a free falling observer with gyroscope and clock can continually perform parallel transport!

In Euclidean space, we know what it means to transport a frame w/o rotating it.
We will need to be able to express a covariant derivative in a particular coordinate system ("curvilinear coords")

\[ x^\alpha \Rightarrow \text{basis vectors} \quad e_\alpha = \frac{\partial}{\partial x^\alpha} \]

In Euclidean (Minkowski) coordinates

\[ \nabla (e_\alpha) = 0 \]

But not so in general coords

Example:

Polar coordinates in the plane

\[ ds^2 = dr^2 + r^2 d\theta^2 \]

\[ \nabla e_r = \nabla e_\theta = 0 \]

\[ \nabla e_r = dx^\beta \Gamma_{\alpha \beta}^r e_\gamma \]

\[ \nabla e_\theta = dx^\beta \Gamma_{\alpha \beta}^\theta e_\gamma \]

\[ \nabla e_r = dr \Gamma_{rr}^r e_r + dr \Gamma_{r\theta}^r e_\theta + d\theta \Gamma_{r\theta}^\theta e_r + d\theta \Gamma_{\theta\theta}^\theta e_\theta \]

\[ \nabla e_\theta = dr \Gamma_{r\theta}^r e_r + dr \Gamma_{\theta\theta}^\theta e_\theta + d\theta \Gamma_{r\theta}^r e_r + d\theta \Gamma_{\theta\theta}^\theta e_\theta \]

\[ \Gamma_{rr}^r = \frac{1}{r} \]

\[ \Gamma_{r\theta}^\theta = -r \]

\[ \Gamma_{\theta\theta} = -r \]

Christoffel:

\[ \Gamma_{\theta\theta} = -\frac{1}{2} g_{\theta\theta}, r = -r \]

\[ \Gamma_{r\theta} = \frac{1}{2} g_{\theta\theta} [g_{\theta\theta}, r] = \frac{1}{r} \]

\[ \frac{d^2 r}{dt^2} = \Gamma_{\theta\theta} (\frac{d\theta}{dt})^2 = -r (\frac{d\theta}{dt})^2 \]

\[ \frac{d\theta}{dt} = \Gamma_{r\theta} \frac{dr}{dt} + \Gamma_{\theta\theta} \frac{d\theta}{dt} \]

\[ \frac{dx^2}{dt^2} = \Gamma_{\alpha \beta}^\gamma \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} \]
using the locally inertial notion of parallel transport:

\[ \nabla_\beta \bar{e}_\gamma = \Gamma^{\alpha}_{\beta \gamma} \bar{e}_\alpha \text{ or } \nabla \bar{e}_\gamma = dx^\beta \Gamma^{\alpha}_{\beta \gamma} \bar{e}_\alpha \]

The Affine connection

\[ \frac{d}{d\lambda} \bar{e}_\gamma = \frac{d}{d\lambda} \Gamma^{\alpha}_{\beta \gamma} \bar{e}_\alpha \]

Warning: \( \Gamma^{\alpha}_{\beta \gamma} \) is not a tensor (does not transform like one)

In particular \( [\Gamma^{\alpha}_{\beta \gamma}]_0 = 0 \) in local flat coordinates

but \( \neq 0 \) in general coordinates

it is telling us something about the coordinates, not actually a geometrical property

In components \( \nabla \rightarrow V^\alpha \rightarrow \)

Two contributions to \( \nabla \bar{V} \)

- how components change along \( \bar{V} \)
- how basis vectors change along \( \bar{V} \) (encoded in the connection)

\[
\frac{d}{d\lambda} (V^\alpha \bar{e}_\alpha) = \left( \frac{d}{d\lambda} V^\alpha \right) \bar{e}_\alpha + V^\alpha \frac{d}{d\lambda} \bar{e}_\alpha
\]

\[ = \left( \frac{d}{d\lambda} (dx^\beta V^\alpha) \right) \bar{e}_\alpha + V^\alpha \frac{d}{d\lambda} (\Gamma^{\alpha}_{\beta \gamma} \bar{e}_\gamma)
\]

\[ = \frac{d}{d\lambda} \left( \frac{d}{d\lambda} V^\alpha + V^\gamma \Gamma^{\alpha}_{\beta \gamma} \right) \bar{e}_\alpha
\]

\[ = (\nabla V)^\beta \bar{e}_\alpha, \text{ where } (\nabla V)^\beta = \frac{d}{d\lambda} V^\alpha + \Gamma^{\alpha}_{\beta \gamma} V^\gamma \]
\[ \nabla \hat{\nabla} = dx^\beta (V^\alpha ; \beta) \hat{e}^\alpha \]

In components we denote the covariant derivative

\[ V^\alpha ; \beta = V^\alpha , \beta + \Gamma^\alpha _{\beta \gamma} V^\gamma \]

Another view:

if \( x_i \)'s are locally flat curves that satisfy \( \nabla \hat{e}_a = 0 \)

where \( \hat{e}_a = \frac{\partial}{\partial y_a} \)

Then \( \nabla (x^a \hat{e}_a) = dy_i \left( \frac{\partial}{\partial y_i} x^a \right) \hat{e}_a \)

So \( \nabla \hat{e}_a = \nabla \left( \frac{\partial}{\partial x^a} \right) = \nabla \left( \frac{\partial y_a}{\partial x^a} \hat{e}_a \right) \)

\[ = dy_i \left( \frac{\partial}{\partial y_i} \frac{\partial y_a}{\partial x^a} \right) \hat{e}_a \quad \text{(Wenzel \( \nabla \hat{e}_a = 0 \))} \]

\[ = dx^\beta \left( \frac{\partial}{\partial x^\beta} \frac{\partial y_a}{\partial x^a} \right) \left( \frac{\partial x^\delta}{\partial y_a} \right) \frac{\partial}{\partial x^\delta} \hat{e}_a = dx^\beta \Gamma^\beta _{\alpha \gamma} \hat{e}_\gamma \]

where

\[ \Gamma^\beta _{\alpha \gamma} = \frac{\partial x^\beta}{\partial y_a} \frac{\partial y_a}{\partial x^\alpha} \frac{\partial y_a}{\partial x^\gamma} \frac{1}{2x^a \partial x^a} \]
We see that $\nabla_{\beta} \alpha$ has $\alpha \leftrightarrow \beta$ symmetry.
- $\Gamma = 0$ in coordinates that differ from $y$'s by a linear transformation.
- Also easy to see how $\nabla_{\beta} \alpha$ transforms.

Covariant derivative of a 1-form

We will define it by demanding that $\nabla$ acting on a tensor obey Leibniz product rule:

$$\nabla (T \otimes S) = \nabla T \otimes S + T \otimes \nabla S$$

We have $\langle dx^\alpha, e_\beta \rangle = \delta^\alpha_\beta$.


\[ \nabla \langle dx^\alpha, e_\beta \rangle = 0 = \langle \nabla dx^\alpha, e_\beta \rangle + \langle dx^\alpha, \nabla e_\beta \rangle \]

But $\nabla e_\beta = dx^\alpha \Gamma^\alpha_\beta e_\alpha$.

\[ \Rightarrow \langle \nabla dx^\alpha, e_\beta \rangle = -dx^\gamma \Gamma^\alpha_\beta \]

or

$$\nabla dx^\alpha = -dx^\gamma \Gamma^\alpha_\beta dx^\beta$$

And so

$$\nabla (w_\alpha dx^\alpha) = dx^\beta (w_\alpha, \beta - \Gamma^\gamma_\alpha_\beta w_\gamma) dx^\alpha$$

or

$$w_\alpha; \beta = w_\alpha, \beta - \Gamma^\gamma_\alpha_\beta w_\gamma$$

WARNING: minus sign.
Now that we know $\nabla dx^\alpha$ and $\nabla e_\beta$, we can covariantly differentiate any tensor $\nabla (T^{\beta} \otimes e_\beta dx^x)$

$$= dx^\alpha \left( T^{\beta\alpha} e_\beta dx^x 
+ T^{\beta\alpha\gamma} \Gamma_{\gamma}^{\beta} e_\delta dx^\delta 
- T^{\beta e_\gamma\Gamma_{\gamma}^{\beta}} e_\delta dx^\delta \right)$$

or $T^{\beta\alpha} = T^{\beta\alpha} + \Gamma_{\gamma}^{\beta} T^{\gamma\delta} - \Gamma_{\gamma}^{\beta} T^{\gamma\delta}$

What is the covariant derivative of $g$?

Claim $\nabla g = 0$. Why?

Because $g$ is used to define parallel transport.

If $\overrightarrow{V}$ and $\overrightarrow{W}$ are parallel transported along $\overrightarrow{U}$

Then $\nabla_{\overrightarrow{U}} \overrightarrow{V} = \nabla_{\overrightarrow{U}} \overrightarrow{W} = 0$

And $g(\overrightarrow{U}, \overrightarrow{W})$: constant along $\overrightarrow{U}$?

$\Rightarrow \nabla g(\overrightarrow{U}, \overrightarrow{W}) = 0 = \nabla g(\overrightarrow{U}, \overrightarrow{W}) + g(\overrightarrow{U}, \overrightarrow{W})$ for any $\overrightarrow{U}$.
More on \( \nabla g = 0 \)

It is a statement about how accurately the Euclidean metric on a tangent plane matches the actual metric.

Because in the locally flat coordinate, we have \((\Gamma = 0)\)

\[ g_{ab,c} = 0 \]

The metric on the tangent plane agrees with the actual metric to linear order in \( \Delta x \)

(Since \( \nabla g \) vanishes in the locally inertial coordinate, and is a tensor \( \Rightarrow \) vanishes in any coords "metric is covariantly constant"")

We can use the property \( \nabla g = 0 \) to express the connection in terms of the metric (makes sense, since as we've seen, metric provides the local notion parallel)

We have:

\[ 0 = \nabla_\alpha g_{\beta\delta} = g_{\beta\delta,\alpha} - \Gamma^\varepsilon_{\alpha\beta} g_{\varepsilon\delta} - \Gamma^\varepsilon_{\alpha\delta} g_{\varepsilon\beta} \]

\[ 0 = \nabla_\beta g_{\alpha\delta} = g_{\alpha\delta,\beta} - \Gamma^\varepsilon_{\delta\alpha} g_{\varepsilon\beta} - \Gamma^\varepsilon_{\delta\beta} g_{\varepsilon\alpha} \]

\[ 0 = \nabla_\delta g_{\alpha\beta} = g_{\alpha\beta,\delta} - \Gamma^\varepsilon_{\delta\alpha} g_{\varepsilon\beta} - \Gamma^\varepsilon_{\delta\beta} g_{\varepsilon\alpha} \]

\[ g_{\beta\delta,\alpha} + g_{\alpha\delta,\beta} - g_{\alpha\beta,\delta} = 2 \Gamma^\varepsilon_{\alpha\beta} g_{\varepsilon\delta} \]

or

\[ \Gamma^\varepsilon_{\alpha\beta} = \frac{1}{2} g^{\varepsilon\delta} \left( g_{\delta\varepsilon,\beta} + g_{\delta\varepsilon,\alpha} - g_{\alpha\beta,\delta} \right) \]
Remark — an alternative derivative \[ \nabla g = 0 \]

Uses expressions for \( g \) and \( \Gamma \) in terms of locally flat coordinates

\[ \Gamma^x_{\alpha \beta} = \frac{\partial^2 y^a}{\partial x^\alpha \partial x^\beta} \]

\[ g_{\alpha \beta} = \gamma_{\alpha \beta} \frac{\partial y^a}{\partial x^\alpha} \frac{\partial y^b}{\partial x^\beta} \]

\[ \Rightarrow g_{\alpha \beta , \gamma} = \gamma_{\alpha \beta} \left( \frac{\partial^2 y^a}{\partial x^\gamma \partial x^\alpha} \frac{\partial y^b}{\partial x^\beta} + \frac{\partial y^a}{\partial x^\alpha} \frac{\partial^2 y^b}{\partial x^\beta \partial x^\gamma} \right) \]

\[ \frac{\partial^2 y^a}{\partial x^\alpha \partial x^\beta} = \Gamma^z_{\beta \alpha} \frac{\partial y^a}{\partial x^z} \Rightarrow \]

\[ g_{\alpha \beta , \gamma} = \Gamma^z_{\beta \alpha} g_{\gamma \beta} + \Gamma^z_{\beta \gamma} g_{\alpha \beta} \Rightarrow \nabla g = 0 \]

Remark: \( \nabla g = 0 \) means we can raise and lower indices either before or after \( \nabla \) act — or contract indices.

Parallel Transport: Connection 1-form

\[ \nabla_v \mathbf{u} = \nabla \mathbf{u} \bigg|_v \]

or in components

\[ 0 = \mathbf{U}^\kappa \nabla_\kappa V^\beta = \mathbf{U}^\kappa \left( \frac{\partial v^\beta}{\partial x^\kappa} + \Gamma^\beta_{\kappa \lambda} v^\lambda \right) \]

Writing \( \mathbf{U}^\alpha = \frac{dx^\alpha}{d\lambda} \), we have

\[ \frac{dv^\beta}{d\lambda} = -\Gamma^\beta_{\kappa \lambda} \frac{dx^\kappa}{d\lambda} v^\lambda \]

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- General convention: comma \( \rightarrow \) semicolon
- Enough freedom in \( \frac{\partial y^a}{\partial x^\alpha} \) to set \( g_{\alpha \beta , \gamma} = 0 \)
This means that as we move from $x$ to $x + dx$, the components of the parallel transported vector $\mathbf{V}$ change according to

$$V^\beta(x+dx) = V^\beta(x) - \Gamma^\beta_\alpha_\delta(x) \ dx^\alpha \ V^\delta(x)$$

$$= \left[ S^\beta_\gamma - (\Gamma^\alpha_\beta(x) \ dx^\alpha)^\gamma \right] V^\gamma(x)$$

In other words, how the components change is described by a matrix

$$\left[ I - \Gamma^\beta_\alpha(x) \ dx^\alpha \right]$$

Here $\Gamma^\alpha_\beta(x)$ is the matrix-valued "connection one-form". It takes tangent vectors to matrices (but remember it is not a tensor).

**Geodesic**

A geodesic is a curve that is locally straight — e.g., the trajectory of a freely falling ("inertial") body. One way to describe it: use the local Minkowski coordinates.
Parameterize the geodesic by \( \lambda \). Convenient to choose \( \lambda \) so geodesic is linear in \( \lambda \) in the locally flat coordinates.

Space-like dx-prop distance
time-like \( \frac{dt}{\lambda} \) prop time
null: \( \lambda \) constant length in local Minkowski coordinate system

In an affine parameterization, a straight path in space-time (in locally flat coords) obeys

\[
0 = \frac{d^2}{d\lambda^2} \gamma^a(\lambda)
\]

and

\[
d\gamma^a = \frac{2\gamma^a}{2\gamma^a \frac{dt}{d\lambda}}
\]

\[
\Rightarrow 0 = \frac{d}{d\lambda} \left( \frac{2\gamma^a}{2\gamma^a \frac{dt}{d\lambda}} \frac{dX^a}{d\lambda} \right)
\]

\[
= \frac{2\gamma^a}{2\gamma^a} \frac{d^2X^a}{d\lambda^2} + \frac{dX^b}{d\lambda} \frac{2\gamma^a}{2\gamma^a} \frac{dX^b}{d\lambda} \frac{dX^a}{d\lambda}
\]

\[
\Rightarrow \frac{d^2X^a}{d\lambda^2} + \frac{dX^b}{d\lambda} \frac{2\gamma^a}{2\gamma^a} \frac{dX^b}{d\lambda} \frac{dX^a}{d\lambda} = 0
\]

\[
\Rightarrow 0 = \frac{d^2X^a}{d\lambda^2} + \frac{\gamma^a}{\gamma^a} \frac{dX^b}{d\lambda} \frac{dX^a}{d\lambda}
\]

Equation of a geodesic in arbitrary coords.

WARNING: only if path is affinely parameterized!
An alternative (more geometrical) viewpoint

The path is locally straight if the tangent to the path at $B$ agrees with the tangent at $A$, parallel transported to $B$

Hence if $\mathbf{v} = \frac{dx}{d\lambda}$ is the tangent to the curve

$$\nabla \mathbf{v} = c(\lambda) \mathbf{v}$$

The affine parameterization is the choice of $t$ along the curve so that the "length" of the tangent remains constant. For this parameterization

$$\nabla \mathbf{v} \cdot \mathbf{v} = 0$$

WARNING: ditto

In components:

$$\mathbf{v} = \frac{dx^\alpha}{d\lambda} \partial_x^\alpha, \quad 0 = (\nabla \mathbf{v} \cdot \mathbf{v})^\beta = v^\alpha \partial_\alpha v^\beta = \frac{dx^\alpha}{d\lambda} \left( \partial_x^\alpha (v^\beta) + \frac{dx^\alpha}{d\lambda} \partial_x^\alpha (v^\beta) \right)$$

$$= \frac{d^2 x^\beta}{d\lambda^2} + \Gamma^\beta_{\alpha \gamma} \frac{dx^\alpha}{d\lambda} \frac{dx^\gamma}{d\lambda}$$
Remark: A geodesic is a curve of extremal length (which we would expect for the "straightest" path).

Proper distance \( L = \int ds = \int_0^1 \sqrt{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} d\lambda \)

Vary with endpoints fixed ⇒
Euler-Lagrange

\[
\frac{d}{d\lambda} \left( \frac{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}{\sqrt{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right)^{\frac{1}{2}} = \frac{2}{\sqrt{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \frac{\partial}{\partial x^\gamma} \left( \frac{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}{\sqrt{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right)
\]

Now choose affine parameterization \( \sqrt{g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} = 1 \)

\( \Rightarrow \quad g_{\alpha\beta,\gamma} \frac{dx^\gamma}{d\lambda} \frac{dx^\beta}{d\lambda} + g_{\alpha\beta} \frac{d^2x^\beta}{d\lambda^2} = \frac{1}{2} g_{\beta\gamma,\alpha} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} \)

or \( g_{\alpha\beta} \frac{d^2x^\beta}{d\lambda^2} + (g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\beta\gamma,\alpha}) \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} \)

This is the geodesic eqn:

\[
\left[ \frac{d^2x^\delta}{d\lambda^2} + \frac{1}{2} \left( 2g_{\beta\gamma,\alpha} - g_{\beta\gamma,\alpha} \right) \frac{dx^\beta, dx^\gamma}{d\lambda} \frac{dx^\delta}{d\lambda} \right]
\]

be careful:

\( g_{\alpha\beta,\gamma} = g_{\beta\gamma,\alpha} \)
There are 3 different types of geodesics:
- Timelike
- Spacelike
- Null

Parallel transport \( g(\frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau}) = \text{constant} \)

(Null geodesics are tangent to the constant)

Since it is constant, timelike remains timelike, etc.
(Freely falling observers always remain on timelike trajectory, and photon, once null, is always null.)

Timelike case:
Set \( t \) as a local max

= Principle of cosmic synchrony

We want the trip to take as long as possible as measured on local clock

where initial and final points are fixed in space time

In special relativity, we should stay at rest — Moving causes time dilation

In GR: clocks run faster higher up, slower lower down (gravitational red shift)
Projectile falls in a parabola --
It is better to stay up high longer -- where my clock would run faster!

Newtonian Limit of Geodesic Equ

Let's look more closely at how to reconcile the geodesic eqn with motion in a Newtonian gravitational field

Newton applies:
- when fields are weak
- when motion is slow

\[ 0 = \frac{d^2 x^i}{dt^2} + \nabla^j \frac{dx^j}{dt} \frac{dx^i}{dt} - \left( \frac{d}{dt} \frac{dx^i}{dt} \right)^2 \]

\( t \) is proper time as affine parameter

\[ \frac{dt}{d\tau} \sim 1 \Rightarrow \frac{dx^i}{d\tau} \sim v \]

\[ \Rightarrow \frac{d^2 x^i}{dt^2} \sim -\Gamma^{i}_{00} \]

We use "nearly minkowski coordinates"

\[ g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad h_{\alpha\beta} \ll 1 \]

\[ \Gamma^i_{00} = \frac{1}{2} g^{ia} (2 g_{00,0} - g_{00,a}) = -\frac{1}{2} h_{00,0} \quad \text{(to linear order in h)} \]
We have \[ \frac{d^2 x^i}{dt^2} = \frac{1}{2} \hbar \cdot \frac{\partial}{\partial x^i} \]

cf. Newton \[ \frac{d^2 x}{dt^2} = -\nabla \varphi \]
Hence \( \hbar \cdot \frac{\partial}{\partial x^i} \)

Hence \( g_{00} = -(1 + 2\varphi) \)

From this, we may infer the gravitational red shift

(Einstein, 1907)

\[ \Delta T^2 = -a_{00} \Delta t^2 \]

Consider 2 clocks in the static field of the Earth.

Proper time is related to coordinate time by

But who cares about coordinate time?

That coordinate time runs at different rates at two different heights has observable consequences.

\[ \Delta T = \sqrt{-g_{00}} \Delta t \]

\[ \sim (1 - \varphi) \Delta t \]

means clock runs "faster" (slower at between ticks) higher up where \( \varphi \) is longer.
\[
\frac{\Delta t_1}{\Delta t_2} = \frac{1 - \phi_1}{1 - \phi_2} = 1 + \phi_2 - \phi_1
\]

\[\sqrt{\frac{g_{22}}{g_{00}(1)}} \quad \text{(1)} \]

\[\Delta t = \frac{\Delta t_2}{\sqrt{g_{00}(1)}}
\]

- If metric is static and sender and receiver are at fixed coordinate positions, this is the same time a travel for each photon
- So photons are received at apart

\[
\frac{\Delta t_1}{\sqrt{g_{00}(1)}} = \Delta t = \frac{\Delta t_2}{\sqrt{g_{00}(1)}}
\]

or \[
\frac{V_2}{V_1} = \frac{\Delta t_1}{\Delta t_2} = \sqrt{\frac{g_{00}(1)}{g_{00}(1)}} = 1 + \phi_1 - \phi_2
\]

- Falling photon detected shifted blue
- Rising photon detected shifted red

\[
(C) \quad \Delta E = -EA\phi \quad \text{Phenomenal kinetic energy as light gains potential energy}
\]

\[(\text{where } E=\hbar \nu)\]
Or. (Einstein 1907):

E. didn't use this argument until 1911.

In accelerating frame:

- and 2 are at rest when photon is emitted.

- when received, 2 is pulling away.

\[ u > v \]

\[ \Rightarrow \text{Red shift} \]

\[ \frac{u}{v} = \frac{\zeta(1-v^2)}{1-\frac{q4h}{c^2}} \]

\[ \text{or} \quad \left[ \frac{du}{v} = -\frac{d\phi}{c^2} \right] \]

- Pound-Rebbka

\[ \Delta h \approx 23 \mu \quad \frac{du}{v} = -\frac{d\phi}{c^2} = \frac{q4h}{c^2} \approx \frac{980 \times 2860}{(3 \times 10^8)^2} = 2.5 \times 10^{-15} \]

Use Monbaeuer and compare with Doppler shift. 

Use Monbaeuer and compare with Doppler shift. 

More on geodesics: Symmetries and conservation laws

If the metric has symmetries

\[ \Rightarrow \text{Constant of the motion along a geodesic} \]

("1st integral" of the geodesic eqn)

Four velocity

\[ u^\alpha = \frac{dx^\alpha}{dt} \]

i.e.

\[ u^\alpha = \frac{dx^\alpha}{dt} \]

and

\[ 0 = \nabla_\alpha u^\alpha = u^\alpha \nabla_\alpha u^\beta \Rightarrow \frac{du^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0 \]
\[
\frac{du^\beta}{dt} = -\Gamma^\beta_{\alpha \gamma} u^\alpha u^\gamma
\]

\[
\frac{1}{2} g^{\beta \delta} (g_{\alpha \delta, \gamma} + g_{\delta \gamma, \alpha} - g_{\alpha \gamma, \delta})
\]

\[
\Rightarrow \quad \frac{du^\beta}{dt} = -\frac{1}{2} (g_{\alpha \delta, \gamma} + g_{\delta \gamma, \alpha} - g_{\alpha \gamma, \delta}) u^\alpha u^\gamma
\]

\[
= -\frac{1}{2} g_{\delta \gamma, \alpha} (\text{because of symmetry})
\]

Okay -- but suppose we write this differently

\[
\Rightarrow \quad 0 = u^\alpha \nabla_\alpha u_\beta = u^\alpha (u_\beta, \alpha - \Gamma^\beta_{\gamma \alpha} u^\gamma)
\]

\[
\Rightarrow \quad \frac{du_\beta}{dt} = \Gamma^\beta_{\gamma \alpha} u_\gamma u^\alpha
\]

\[
= \frac{1}{2} g^{\delta \gamma} (g_{\beta \delta, \alpha} + g_{\delta \alpha, \beta} - g_{\beta \alpha, \delta}) u_\gamma u^\alpha
\]

\[
= \frac{1}{2} (g_{\beta \delta, \alpha} + g_{\delta \alpha, \beta} - g_{\beta \alpha, \delta}) u^\delta u^\alpha
\]

Another useful gauge form of geodesic eqn

Conservation Law:

If \( g_{\beta \gamma} (x) \) is independent of \( x \),

then \( u_\gamma = \text{constant along geodesic} \)

WARNING: \( u_\gamma \) is constant but \( u^\gamma \) is not!
\( X^r = c \text{r} \) \( \Rightarrow \)

\[ u_\gamma = u \cdot \tilde{e}^\gamma = \tilde{u} \cdot \tilde{e}^\gamma = \text{const} \]

Another, more general "conservation law"

\( u \cdot \tilde{u} = -1 = \text{constant} \) \( (\text{timelike}) \)

\( = 0 = \text{constant} \) \( (\text{null}) \)

\( \text{i.e.,} \quad \nabla_\tilde{u} (u^\gamma \tilde{u}_\gamma) = 2 \tilde{u} \cdot (\nabla_\tilde{u} \tilde{u}^\gamma) = 0 \)

(Same for 4-momentum \( u^\gamma \rightarrow \) vanishes along geodesic)

Example: Station (\( t \)-independent) metric

\[ \Rightarrow u_0 \quad (\text{or } \rho_0) = \text{constant} \]

constant \( = \frac{1}{2} g_{00}(u_0)^2 + g_{ij} u^i u^j \)

\[ -1 = g_{00}(u_0)^2 + g_{ij} u^i u^j \]

E.g., \( g_{33} \left( \frac{dx^3}{dx^0} \right)^2 = -1 + \left( \frac{1}{-g_{00}} \right) u_0^2 \)

should correspond to conservation of energy in Newtonian field

(\( \text{cf. Example 24.2 in B+T} \))

\( \text{Here } g_{00} \text{ can depend on } x \)

\( \text{what is the more general form of } \Gamma \text{ in } \Gamma \text{?} \)