[1] A Cooling Hamburger - 30 pts

(a) [5 pts] Express $T_0$ in terms of $W$ and the Stefan-Boltzmann constant.
A perfect black body in thermal equilibrium implies:

$$ W = 4\pi R^2 \sigma_B T_0^4 $$ (1.2)

(b) [10 pts] turn off microwave; Hamburger cools due to radiation only; temperature of Hamburger is uniform;

$$ C = \frac{dU}{dT} = cV $$ (1.3)

The Hamburger looses energy due to radiation $dU_{rad}$:

$$ dU_H = cVdT = -dU_{rad} $$ (1.4)
$$ dU_{rad} = 4\pi R^2 \sigma_B T(t)^4 dt = A\sigma_B T(t)^4 dt $$
$$ \Rightarrow cVdT(t) = -A\sigma_B T(t)^4 dt $$

$$ \Rightarrow \frac{dT}{dt} = -\frac{A\sigma_B}{cV}T^4 = -\frac{3A\sigma_B}{cR}T^4 $$ (1.5)

Solving the differential equation by separation of variables:

$$ \frac{dT}{T^4} = -\frac{A\sigma_B}{cV}dt $$ (1.6)
$$ \Rightarrow \int_{T_0}^{T} \frac{dT'}{(T')^4} = -\frac{A\sigma_B}{cV}t $$
$$ \frac{1}{3} \left[ \frac{1}{T^3} - \frac{1}{T_0^3} \right] = -\frac{A\sigma_B}{cV}t $$ (1.7)

$$ \Rightarrow T = \left[ \frac{1}{T_0^3} + \frac{3A\sigma_B}{cV}t \right]^{-\frac{1}{3}} $$ (1.8)

(c) [5 pts] How long does it take the Hamburger to cool from temperature $T_0$ to $T_0/2$?
Using eq.(1.7):

$$ \frac{8}{T_0^3} = \frac{1}{T_0^3} + \frac{3A\sigma_B}{cV}t_{1/2} $$ (1.9)
$$ \Rightarrow t_{1/2} = \frac{7cV}{3A\sigma_B T_0^3} = \frac{7cR}{9\sigma_B T_0^3} $$ (1.10)
(d) [10 pts] Model the paper as perfectly absorbing shell around the Hamburger.
Technically, we have to set up a rate equation for the burger and the paper.

\[ dU_H = -A\sigma_B T_H^4 dt + A\sigma_B T_p^4 dt \]  

(1.11)

The energy balance for the wrap paper is given by:

\[ dU_p = A\sigma_B T_H^4 dt - 2A\sigma_B T_p^4 dt, \]

(1.12)

where the factor of two in eq.(1.12) accounts for the fact that the paper radiates in- and outwards. Taking into account that the paper has negligible heat capacity \( C_p \Rightarrow dU_p = C_p dT \approx 0 \), we find:

\[ T_p = 2^{-1/4} T_H. \]  

(1.13)

Using this in the first equation it is straight forward to show that:

\[ cV dT_H = -\frac{1}{2} A\sigma_B T_H^4 dt. \]  

(1.14)

We see that the power loss of the Hamburger due to the paper is reduced by a factor of \( \frac{1}{2} \) in comparison to our analysis in part (c). The computation of the cooling time \( t_{1/2} \) follows in complete analogy to (c) and one finds a doubling of \( t_{1/2} \).

\[ t_{1/2} = \frac{14cV}{3A\sigma_B T_0^3} = \frac{14cR}{9\sigma_B T_0^3} \]  

(1.15)
[2] Pressure dependence of adsorption - 30 pts

Show that the probability that the adsorption site is occupied by a molecule has the form

\[ f = \frac{p}{p + p_0(\tau)}, \]  

(2.1)

and express \( p_0(\tau) \) in terms of \( \tau, \epsilon \) and \( m \).

**Note**: this exercise is in one to one correspondence with the example treated in Kittel & Kroemer, pg. 141.

The adsorption site may be occupied by 0 or 1 molecules. In this case the Gibbs sum is given by:

\[ Z = 1 + \lambda \exp \left[ -\epsilon / \tau \right]. \]  

(2.2)

If energy must be added to remove the molecule from the adsorption site, then \( \epsilon < 0 \). In our case the molecule gains energy as it binds to the adsorption site so that \( \epsilon > 0 \).

In diffusive equilibrium, the activities of the adsorption site and the surrounding gas have to be equal. For an 3D ideal gas, the activity \( \lambda \) is given by:

\[ \lambda = \frac{n}{n_Q} = \frac{p}{\tau n_Q} \]  

(2.3)

with quantum concentration

\[ n_Q = \left( \frac{m \tau}{2 \pi \hbar^2} \right)^{3/2}. \]  

(2.4)

The occupation fraction is given by:

\[ f = \frac{\lambda \exp \left[ -\epsilon / \tau \right]}{1 + \lambda \exp \left[ -\epsilon / \tau \right]} \]  

(2.5)

\[ = \frac{1}{1 + \lambda^{-1} \exp \left[ \epsilon / \tau \right]} \]  

\[ = \frac{p}{p + n_Q \tau \exp \left[ \epsilon / \tau \right]} . \]  

(2.6)

Comparing this with eq.(2.1), we find:

\[ p_0(\tau) = n_Q \tau \exp \left[ \epsilon / \tau \right] = \left( \frac{m \tau}{2 \pi \hbar^2} \right)^{3/2} \tau \exp \left[ \epsilon / \tau \right] . \]  

(2.7)
[3] Surface of a two-dimensional liquid - 40 pts

(a) [10 pts] Show that \( N_+ \) and \( N_- \) can be expressed in terms of \( N_0 \) and \( L \).

Since we assume that the left and right points are at the same height, we have the constraint:

\[
N_+ = N_-.
\] (3.1)

Taking the length \( L \) between left and right end of the liquid into account, we obtain:

\[
L = N_0 + \frac{N_-}{2} + \frac{N_+}{2}
\] (3.2)

\[
\Rightarrow N_+ = N_- = L - N_0.
\] (3.3)

Plugging this into the formula for the energy

\[
E = \epsilon (N_+ + N_- + N_0 - L) = \epsilon (L - N_0)
\] (3.4)

For later convenience we note that

\[
\frac{E}{\epsilon} = L - N_0.
\] (3.5)

(b) [10 pts] Multiplicity function \( g(N_0, L) \).

\[
g(N_0, L) = \frac{(N_+ + N_- + N_0)!}{N_+!N_-!N_0!} = \frac{(2L - N_0)!}{[(L - N_0)!]^2 N_0!}
\] (3.6)

(c) [10 pts] Entropy as function of energy and definition of temperature.

Taking into account the multiplicity function eq.(3.6) and the definition of the entropy in the microcanonical ensemble \( \sigma = \log g(N_0, L) \), we have:

\[
\sigma = \log(2L - N_0)! - 2 \log [(L - N_0)!] - \log N_0!
\] (3.8)

\[
\approx (2L - N_0) \log(2L - N_0) - 2(L - N_0) \log(L - N_0) - N_0 \log N_0
\]

Expressing everything in terms of energies:

\[
\sigma \approx \left( L + \frac{E}{\epsilon} \right) \log \left( L + \frac{E}{\epsilon} \right) - 2 \frac{E}{\epsilon} \log \left( \frac{E}{\epsilon} \right) - \left( L - \frac{E}{\epsilon} \right) \log \left( L - \frac{E}{\epsilon} \right)
\] (3.9)

Using the definition of temperature \( \frac{1}{\tau} = \frac{\partial \sigma}{\partial E} \) and the result for \( \sigma \) above, we obtain:

\[
\frac{1}{\tau} = \frac{1}{\epsilon} \left[ \log \left( L + \frac{E}{\epsilon} \right) + \log \left( L - \frac{E}{\epsilon} \right) - 2 \log \left( \frac{E}{\epsilon} \right) \right] + \left[ \frac{1}{\epsilon} + \frac{1}{\epsilon} - \frac{2}{\epsilon} \right]
\]

\[
= \frac{1}{\epsilon} \log \left[ \frac{L^2 \epsilon^2 - E^2}{E^2} \right]
\] (3.10)

A useful representation of the result is given by:

\[
e^{\epsilon/\tau} = \frac{L^2 \epsilon^2 - E^2}{E^2}
\] (3.11)
(d) [10 pts] Solve for $E(\tau)$ and find the length of the surface $l(\tau)$ as function of temperature $\tau$. Discuss the high and low temperature limits $\tau \gg \epsilon$ and $\tau \ll \epsilon$.

$$l(\tau) = N_0 + N_+ + N_- = N_0 + 2L - 2N_0 = L + \frac{E(\tau)}{\epsilon}, \quad (3.12)$$

Solving eq.(3.11) for $E(\tau)$ yields:

$$E(\tau) = \frac{L\epsilon}{\sqrt{1 + \exp[\epsilon/\tau]}} \quad (3.13)$$

In the high temperature limit $\tau \gg \epsilon$ we approximate:

$$e^{\epsilon/\tau} \approx 1 \Rightarrow E(\tau) \approx \frac{L\epsilon}{\sqrt{2}}$$

$$l(\tau) \approx L(1 + \frac{1}{\sqrt{2}}) \quad (3.14)$$

In the low temperature limit $\tau \ll \epsilon$:

$$E(\tau) \approx L\epsilon e^{-\epsilon/2\tau} \rightarrow 0$$

$$l(\tau) \approx L \quad (3.15)$$

which corresponds to a flat surface as expected.