4. Thermal Radiation and Planck Distribution

Read: Chapter 4  (May skip "electrical noise" section)

Do: Prob 1, 2, 4, 5, 6, 7, 8

The Planck Distribution

In quantum theory, light consists of particles. Einstein's light quantum hypothesis (1905) was not assumed by Planck (1900), and not widely accepted until 1920.

\[ E_{\text{photon}}(\nu) = h\nu \]
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In a (finite size) cavity, only certain (discrete) \( \nu \)'s are allowed (see below) for a cavity in thermal contact with reservoir (temp \( T \)) how many photons at frequency \( \nu \)?

Prob distribution

\[ P_{\nu}(s) = \frac{1}{Z_{\nu}} e^{-s\hbar\nu/kT} \quad (\text{Boltzmann factor}) \]

Use

\[ \sum_{s=0}^{\infty} x^{s} = \frac{1}{1-x} \quad \Rightarrow \quad Z_{\nu} = \frac{1}{1-e^{-\hbar\nu/kT}} \]

So

\[ S_{\nu} = P_{\nu}(s) = e^{-\hbar\nu/kT} (1 - e^{-\hbar\nu/kT}) \]
Find expectation value of occupation number for photon mode with frequency \( \omega \):

\[
\langle S \rangle_\omega = \sum_{s=0}^{\infty} s P_s(\omega)
\]

Note:

\[
\sum_{s=0}^{\infty} s e^{-\frac{s \omega}{kT}} = -\frac{d}{dY} \sum_{s=0}^{\infty} e^{-sy}, \quad Y = \frac{\omega}{kT}
\]

\[
= -\frac{\frac{d}{dY}}{Y} (1 - e^{-Y})^{-1}
\]

\[
= \frac{e^{-Y}}{(1 - e^{-Y})^2}
\]

and:

\[
\langle S \rangle_\omega = (1 - e^{-Y}) \left( \sum_{s=0}^{\infty} s e^{-sy} \right) = \frac{e^{-Y}}{2 - e^{-Y}}
\]

\[
= \frac{1}{e^{\frac{\omega}{kT}} - 1}
\]

- The Planck distribution function.

**Limits:**

\[
\langle S \rangle_\omega = \frac{1}{e^{\frac{\omega}{kT}} - 1} \Rightarrow \frac{e^{-\frac{\omega}{kT}}}{\omega/kT} \quad \omega \to 0
\]

\[
\langle S \rangle_\omega = \frac{\hbar \omega \langle S \rangle_\omega}{e^{\frac{\hbar \omega}{kT}} - 1} \Rightarrow \frac{\hbar \omega e^{-\frac{\hbar \omega}{kT}}}{\hbar \omega/kT} \quad \omega \to \infty
\]

(Thus the "classical" or \( \omega \to 0 \) limit; typical energy stored in a mode is temperature \( T \) - a formula with no \( \hbar \) appearing.)
These formulas hold for each mode of radiation in a cavity. To find, e.g., the energy of a photon goes in thermal equilibrium in a cavity, we must sum over the modes.

E.g., suppose the cavity is a cubic box of side L, and that electric field $E_{ll}$ vanishes on the cavity walls (conducting cavity).

Solve wave equation \( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \) \( E = 0 \)
by expanding in normal modes (Fourier analysis).

\[
E_x = E_{x(0)} \cos(k_x x) \sin(k_y y) \sin(k_z z) e^{i \omega t} \\
E_y = E_{y(0)} \sin(k_x x) \cos(k_y y) \sin(k_z z) e^{i \omega t} \\
E_z = E_{z(0)} \sin(k_x x) \sin(k_y y) \cos(k_z z) e^{i \omega t}
\]

Then

\[
k_x L = \pi N_x \\
k_y L = \pi N_y \\
k_z L = \pi N_z
\]

This solves wave eqn \( \omega^2 = c^2 (k_x^2 + k_y^2 + k_z^2) = c^2 \kappa^2 \)

In addition, we must have

\[ \nabla \cdot \vec{E} = 0 \text{ or } \vec{K} \cdot \vec{E}^{(0)} = 0 \]

\[ \vec{E}^{(0)} \rightarrow \vec{K} \]

2 polarizations.
Now since
\[ \mathbf{K} = \frac{L}{2} (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z) \]
the sum over allowed values of \( \mathbf{K} \) can be replaced by an integral as \( L \to \infty \)
\[ \sum_{\mathbf{K}} \to 2 \left( \frac{L}{\pi} \right)^3 \int_0^\infty dK_x \int_0^\infty dK_y \int_0^\infty dK_z \]
(from counting polarizations)
And frequency \( \omega = c^2 K^2 \)
depends only on \( K_x, K_y, K_z \), so
\[ S \mathbf{K} \to \frac{1}{8} 4\pi \int_0^\infty dK K^2 \]
so \[ \sum_{\mathbf{K}} = 2 \left( \frac{L}{\pi} \right)^3 \frac{1}{8} 4\pi \int_0^\infty dK K^2 \]
and e.g. \( U = \sum_{\mathbf{K}} \frac{\varepsilon \omega}{\varepsilon \omega + 1} \)
\[ = \frac{L^3}{\pi^2} \int_0^\infty dK K^2 \frac{\varepsilon \omega}{\varepsilon \omega + 1} \]
Let \( x = \varepsilon \omega / \kappa \)
\[ = \frac{L^3}{\pi^2} \left( \frac{\varepsilon}{\kappa c} \right)^4 \varepsilon \kappa \int_0^\infty dx x^3 \frac{x^3}{e^x - 1} \]
Volume \( V = L^3 \)
\[ \text{dimensionless integral} \]
\[ = \frac{\pi^4}{15} \]
\[ U/V = \frac{\pi^2}{15} \left( \frac{hc}{k} \right)^3 \]  

Stefan-Boltzmann radiation law

\[ \frac{U}{V} = \int_0^\infty \omega^3 \frac{k}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar \omega/kT} - 1} \]  

\( \omega = \text{spectral density} \)  

For the Planck black-body spectrum of radiation

\[(\text{Peak at } x = \frac{k\omega}{c} \approx 2.82)\]

\[ h = 6.582 \times 10^{-16} \text{ eV sec} \]
\[ c = 2.998 \times 10^{10} \text{ cm sec}^{-1} \]
\[ k = 1.1605 \text{ eV K}^{-1} \]
\[ \lambda = 2.725 \text{ K} \]
\[ \omega = 3.35 \times 10^7 \text{ Hz} \]
\[ \lambda \approx 2.34 \times 10^{-11} \text{ eV} \]

In the low frequency limit, the spectral density behaves like

\[ \omega = \frac{\omega^2}{\pi^2 c^3} \]  

For the sun

- Peak in 880 nm
- Main emission yellow in 570-590 nm
- Sun should be white if hot 6500 K (shifted red by scattering, which makes sky blue)
- Purely classical (not T) - If not persisted for all T
We would have
\[ \frac{d}{dV} \int d\omega \, \omega \, = \infty \]

Quantum physics intervenes to prevent this unphysical catastrophe.

**Black Body**

Suppose we cut a small hole in a cavity. At low temperature, the hole is black:
- Radiation incident on it is absorbed by the cavity.
- If cavity is hot, radiation is emitted through the hole.

\[ T = \frac{300^\circ K}{1 \text{Katal}} \times 0.258 \text{ eV} \approx \frac{1}{38.7} \text{ eV} \]

\[ e^{-30} \approx 10^{-13} \]

What is radition energy flux (rate of emission per unit time and area)?

Radiation gas is isotropic — photon velocities are uniformly distributed in solid angle (and speed = c at all frequencies).

\[ \text{i.e. energy density due to photons with velocity in direction } \mathbf{n} \text{ is} \]

\[ U/V \frac{d\Omega}{4\pi} \]

In time interval \( dt \), quanta are emitted by the hole if within

\[ (c dt) \cos \theta \] of edge of the cavity.

Energy due to such quanta is

\[ d(\text{Energy}) = U/V (\text{Area}) c (dt) \cos \theta \frac{1}{2} d\cos \theta \]
\[ J = \text{Flux} = \frac{\text{Energy}}{\text{Area} \times \text{dt}} = \frac{U}{\lambda} \int_0^L \sin \theta \cos \theta \, d\theta \]

\[ = \frac{U}{\lambda} \frac{1}{4} \pi \]

So we have

\[ J = \frac{\pi^2}{60} \frac{1}{h^2} \frac{1}{T^4} \]

\[ b_8 = \text{the Stefan-Boltzmann constant} \]

Rate at which energy is lost through a hole in a cavity — in equilibrium with a surface at temperature \( T \) (per unit area)

A block-body is one that absorbs all radiation incident on its surface (does not reflect)

Any block-body radiates like a hole in a cavity at the same temperature

To see this — imagine closing the hole by covering it with a surface of the block body. If cavity and body are

in thermal equilibrium at temperature \( T \), then body will emit radiation at the same rate as the hole.

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- Thermal contact
- Enclosed body
- Cavity body
- Cavity
- Direction of radiation hemisphere
Non-black body

Absorption = \( a \cdot \text{ (black body absorption) } \)
\( a = \text{absorptivity} \)

Emission = \( e \cdot \text{ (black body emission) } \)
\( e = \text{emissivity} \)

Claim: \( a = e \)

Why? Consider such a body at temp \( T \) in equilibrium with radiation in a cavity. It must then emit and absorb the same rate.

We could also cover the body with a filter that transmits radiation in frequency range \( (\omega, \omega + \Delta \omega) \) and emits all else then see that

\[
\frac{a(\omega)}{\omega} = \frac{e(\omega)}{\omega}
\]

-- Kirchhoff's Law

Historical Notes by Abraham Pais

Kirchhoff (1859) was the first to realize that the spectral density of radiation in equilibrium with a body has universal character?

\[
\frac{U}{V} = \int_{\omega} a(\omega) \omega d\omega
\]

where \( a(\omega, T) \) is a universal function independent of the constitution of the body.
So it was recognized that this poses a fundamental problem — what is this function $\varphi(\lambda)$?

Stefan conjectured in 1879, and Boltzmann proved in 1884 for a black body

$$\frac{U}{V} = \int_0^\infty \varphi(\lambda) \lambda^4 d\lambda = \text{const} \cdot T^4$$

Proof uses thermodynamics and Maxwell radiation theory. Challenge — what is this constant of proportionality?

In 1893, Wien extended these arguments to show

$$\varphi(\lambda) = \kappa^3 f(\lambda/T)$$

— as far as one could go with classical thermodynamics.

Become an experimental priority to measure $\varphi(\lambda)$. (Need to measure for infrared — an experimental challenge)

Rubens-Kurlbaum present their data on oct 25, 1900 — fit by a curve found by Planck

$$\varphi(\lambda) = \frac{k}{\pi^2 c^3} \lambda^3 e^{kT/\lambda}$$

(For exponential until for large $\lambda$ had been anticipated by Wien in 1896.)