Ferromagnetism

Recall, a magnetic field applied to a magnetic moment tends to align the moment with the field (exerts torque)

\[ B \uparrow \rightarrow \vec{M} \]  \[ E = -\vec{B} \cdot \vec{M} \]  (minimize energy by lining up)

\text{Current loop spin}

So current loops reinforce the applied field

\text{Magnetization} \quad \vec{M} = \text{magnetic moment per unit volume}

\[ \vec{M} = \chi_{m} \vec{H} \]  \text{applied field}

\text{\( \chi_m \)} = \text{magnetic susceptibility}

So total magnetic field given by

\[ \vec{B} = \vec{H} + 4\pi \vec{M} \]

\text{averaged field}

\text{in inhomogeneous}

\[ \vec{B} = \vec{H} + 4\pi \chi_{m} \vec{H} = \mu \vec{H} \]

\( \mu \) = magnetic permeability

Linear relation actually applies only to weak applied field

\[ \vec{M} \rightarrow \vec{H} \]  \text{Paramagnetism}

\[ \text{Magnetic moment aligns with } \vec{H} \]

but turns off when \( H \rightarrow 0 \)

\text{In some magnetic materials (e.g., iron), at sufficiently low temperature,}
\text{M remains nonzero as } H \rightarrow 0 \text{ (c.77 K) weak ferromagnetism)}

\text{In ferromagnetism -- spontaneous magnetization)}
M, "spontaneous mag"

\[ H \]

For \( M \) as discontinuous as a function of \( H \)

- Example 1st order phase transition

(Analogous to jump in volume at given pressure in gas-liquid transition)

in practice - metastability. Real magnets exhibit hysteresis

Claim: \( H \) analogous to pressure

\[ M(\text{volume}) \] analogous to volume

\[ X_p = \left( \frac{\partial M}{\partial H} \right)_T \] analogous to compressibility

strongen this analogy by considering "magnetic "

\[ p dV = (\text{vol}) \times H dM \]

- Apply \( H \) by putting material in a solenoid

\[ H = \frac{\mu_0 n I}{L} \]

\[ \text{curl } E = -\frac{1}{c} \frac{\partial B}{\partial t} \Rightarrow V = \frac{1}{c} \frac{\partial \Phi}{\partial t} (\text{flux}) \]

\[ V = \frac{1}{2} \mu_0 n \text{Area} \frac{\partial B}{\partial t} \] (Faraday)

Power \[ P = \frac{dW}{dt} = IV = \frac{\mu_0}{4\pi} n \cdot H \cdot \frac{1}{2} \mu_0 \text{Area} \frac{\partial B}{\partial t} = \frac{\mu_0}{4\pi} H \frac{\partial B}{\partial t} \]
\[ dW = \frac{V}{4\pi} H dB = \frac{V}{4\pi} H d(H + M) \]

\[ = -V \left( \frac{1}{8\pi} dH^2 + H dM \right) \]

\[ \text{unchanged} \]
\[ \text{sample field} \]

\[ dW = V(HdM) \]

\[ H, \text{like p, is applied} \]
\[ M, \text{like V, no response} \]
\[ X, \text{like K, tells how stiff it responds} \]

\[ \text{Heat up -- how do materials behave?} \]

\[ \begin{array}{c}
M \\
\uparrow \\
\rightarrow \\
H \\
\rightarrow \\
C \\
\uparrow \\
\rightarrow \\
P
\end{array} \]

\[ \text{Behavior of } \tau \rightarrow \theta \]

\[ \begin{array}{c}
M, \text{phase} \\
\nearrow \\
\rightarrow \\
T \\
\rightarrow \\
C \\
\nearrow \\
\rightarrow \\
\tau _{c}
\end{array} \]

\[ \text{values of } V \text{ are in mixed phases.} \]

\[ \begin{array}{c}
A \\
\uparrow \\
\rightarrow \\
\tau _{c} \\
\rightarrow \\
1^{st} \\
\rightarrow \\
2^{nd}
\end{array} \]
We want to develop analog of Van der Waals (1873) 
Theory (Pure Weiss - 1907)

First - single model of paramagnetism 
(alignment of spins by applied field) 
(same has magnetic moment \( \mu \))

\[ \Delta E = 2\mu B \]

Boltzmann \[ \frac{\nu_{\uparrow}}{\nu_{\downarrow}} = e^{-\frac{2\mu B}{kT}} \]

Factorial

\[ \frac{\nu_{\uparrow} \cdot \nu_{\downarrow}}{\nu_{\uparrow} + \nu_{\downarrow}} = \frac{1 - e^{-2\mu B/kT}}{1 + e^{-2\mu B/kT}} = \tanh \left( \frac{\mu B}{kT} \right) \]

So...

if \( N = \text{no./volume of spins} \), we have

\[ M = \nu_{\uparrow} \tanh \left( \frac{\mu B}{kT} \right) \]

Here \( B \) is the magnetic field seen by a spin which depends on magnetization as well as applied field.

Model: \[ B_{\text{eff}} = H + \frac{1}{M} \]

(mentioned as “mean field theory”)

So - relation between \( H \) and \( M \) becomes non-linear

\[ M = \nu_{\uparrow} \tanh \left[ \frac{\mu B}{kT}(H + 1/M) \right] \]

How do isocurves behave?
Look at $H = 0$ (turn off applied field)

Let $m = \frac{M}{\eta \mu}$

$m = \tanh \left( \frac{\eta \mu^2}{2} \right)$

either one or three solutions, depending on slope at the origin.

For $T < T_c$, isotherm looks like $M$

Unit $Z$ Maxwell construction tells us there is a 1st order phase transition at $H = 0$.

Solutions with instantaneous magnetization turn on for

$$\frac{\eta \mu^2}{2} > 1$$

or $T < T_c = \frac{\eta \mu^2}{2}$


\[ \text{coexistence phases} \]
We have spontaneous magnetization ($H=0$)

$$M \rightarrow T$$

Critical Isotherm

$$\frac{dM}{dH} = \chi_m \rightarrow \infty$$

divergent susceptibility

$$A \rightarrow 0$$

$\langle c \rangle$ is a second order phase transition ends $T_c$.

**Critical Exponents**

How does magnetization turn on?

We have $M = \tanh \left( \frac{\mu}{T} m + \frac{m}{T} H \right)$

Rescale temperature $\tilde{T} = T/T_c$ = $T/\mu m$

Consider $H = 0$

$$\Rightarrow M = \tanh \left( \frac{m}{T} \right)$$

For $T$ near $T_c$, $\tanh$ is $\approx m < 0$, and we can expand

$$\tanh x \approx x - \frac{1}{3} x^3 + \cdots$$

$$\Rightarrow M \approx \frac{m}{T_c} - \frac{1}{3} \frac{m^3}{T_c^3}$$
Write \( \hat{\tau} = 1 + \delta \hat{\tau} \)

\[
\Rightarrow \quad m(1 - \frac{2}{1 + \delta \hat{\tau}}) = -\frac{1}{3} \frac{m^3}{1 + \delta \hat{\tau}} + \cdots
\]

\[
\Rightarrow \quad m(\delta \hat{\tau}) \approx -\frac{1}{3} m^3
\]

\[
\Rightarrow \quad \hat{m}^2 \approx 3 \left( \frac{\tau - \tau_c}{\tau_c} \right)
\]

So \( \hat{m} \sim (\tau_c - \tau)^{\frac{1}{2}} \) - as in Van der Waals theory.

How can magnetic susceptibility blow up?

Consider slope of isotherm near critical.

Write \( \hat{m} = \tanh \left( \frac{m}{\hat{\tau}} + \frac{m}{\tau_c} H \right) \)

and use \( \tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B} \)

\[
\Rightarrow \quad \hat{m} = \frac{\tanh \left( \frac{m}{\hat{\tau}} \right) + \hat{m}}{1 + \tanh \left( \frac{m}{\hat{\tau}} \right)}, \text{ where } \hat{m} = \tanh \left( \frac{m}{\tau_c} H \right)
\]

\[
\Rightarrow \quad \hat{m} = \frac{m - \tanh \left( \frac{m}{\hat{\tau}} \right)}{1 - m \tanh \left( \frac{m}{\hat{\tau}} \right)}
\]

Now we use

\[
X_c = \left( \frac{\partial \hat{m}}{\partial H} \right)_T = \left( \frac{\partial m}{\partial \hat{\tau}} \right)_T \left( \frac{\partial \hat{\tau}}{\partial H} \right)_T, \text{ and hence } \left( \frac{\partial \hat{m}}{\partial H} \right)_T \hat{m} \hat{\tau} = 1 + \delta \hat{\tau}
\]
Expand \( \tanh x = x - \frac{1}{3} x^3 \)

\[
\Rightarrow \quad h = \left[ m - \frac{m^3}{3} + \frac{1}{3} \left( \frac{m^2}{t} \right)^3 \right] \left[ 1 + \frac{m^2}{t} \right]
\]

\[
= m \left( 1 - \frac{1}{t} \right) + m^3 \left( \frac{1}{3} \frac{1}{t^3} + \frac{1}{t} \left( 1 - \frac{1}{t} \right) \right)
\]

\[
\approx m \left( \delta t \right) + \frac{1}{3} m^3
\]

Hence, \( \ln \tau > \tau_c, \quad m^2 = 0 \),

we have \( \frac{\partial h}{\partial m} = \delta t \approx \tau - \tau_c \)

\(-\ln \tau < \tau_c, \quad m^2 = -3 \delta t \)

\[
\frac{\partial h}{\partial m} = \delta t - \delta t \approx -3 \delta t = 2 \frac{\tau_c - \tau}{\tau_c}
\]

Also, \( M = \eta m m \Rightarrow \frac{\partial M}{\partial m} = \eta m \)

\[ h = \tanh \left( \frac{\tau - \tau_c}{\tau_c} \right) \Rightarrow \frac{\partial h}{\partial \tau} \approx \frac{1}{\tau_c} \]

\[
\Rightarrow \quad \chi_m = \frac{\eta m^2}{\tau_c} \times \begin{cases} 
(\tau - \tau_c)^{-1} \quad \tau > \tau_c \\
\frac{1}{2} \left( \frac{\tau_c - \tau}{\tau_c} \right)^{-1} \tau < \tau_c
\end{cases}
\]

(weak-field susceptibility

for \( \lambda = 0 \))

\[
\chi \uparrow \quad \frac{1}{\tau_c} \quad \tau_c
\]

---

the same exponents

and formulas \( T \) and \( \lambda \) as in van der Waals theory.
Experiment:

\[ M \sim (T_c - T)^\beta \quad \beta \sim 0.33 \quad \text{(for \theta > 0)} \]

\[ X_m \sim (T - T_c)^\delta \quad \delta \sim 1.3 \]

Lands' Theory of Phase Transitions

- A systematic and general approach to mean-field theory.

Usually, we evaluate the energy, e.g., \( F(M, \theta) \) for most possible configurations (equilibrium). But we can also consider \( F = U - TS \) for other configurations (as we did implicitly in order to arrive at the Maxwell construction). Then equilibrium values are found by minimizing \( F \), e.g.,

\[ F(M, \theta) \] can be minimized for fixed \( \theta \) to find equilibrium value \( M_0 \).

Lands' idea is that \( F(M, \theta) \) is actually an analytic function that can be expanded as a power series in \( M, \theta \), and any other arguments, but the equilibrium value of \( F \) may still
be unstable, because the freezing temperature may be unstable in temperature.

Denote by $E$ the order parameter for the transition — which is to take the value in equilibrium

$$E_0 = 0 \quad \text{for } T > T_c$$

$$E_0 \neq 0 \quad \text{for } T < T_c$$

Eq. $E = M$ or $E = (\text{liquid} - \text{gas})$

called order parameter because

$E_0 = 0$ in a system with no long-range order

$E_0 \neq 0$ in ordered phase

(All spins favor pointing the same way)

We'll assume system has $E \to -E$ symmetry (good assumption for a system)

This means $E$ is an even function of $E$

So $E_0 = 0$ respects the symmetry, but $E_0 \neq 0 \sim \text{breaks}$ it

There will be two states $\pm E_0$ with same free energy — equally good as equilibrium states — spontaneous symmetry break.
we assume \( F \) can be expanded in
powers of \( \xi \)

\[
F(\xi, \eta) = g_0(T) + \frac{1}{2} g_2(T) \xi^2 + \frac{1}{4} g_4(T) \xi^4 + \frac{1}{6} g_6(T) \xi^6 + \ldots
\]

Note: Landau's theory does not really require this expansion to be convergent; it is good enough for it to be a reasonable (asymptotic) approximation carried out to, e.g., order \( \xi^4 \).

Now—Landau says—Suppose that

\[
g_2(T) \rightarrow 0 \quad \text{as} \quad T \rightarrow T_c
\]

with

\[
g_2 > 0 \quad T > T_c
\]
\[
g_2 < 0 \quad T < T_c
\]

Suppose also that \( g_4(T_c) > 0 \).

Then:

\[
\xi_0 = 0
\]

The order parameter \( \xi \) turns on as we lower temperature through \( T_c \)

\( \Rightarrow \) phase transition.
For $T < T_c$, imagine coupling an external field $\phi$ to $\phi$

$$F(\phi, \phi) \rightarrow F(\phi, \phi) - \lambda \phi$$

(Find $G(\lambda, \phi)$ by minimizing w.r.t $\lambda$)

Extremum at $\lambda = \frac{\partial F}{\partial \phi}$

There is an instability in region where $F$ is concave down.

We observe two well defined phases (the free energy of a homogeneous phase, and the mixed phase has lower free energy, and is stable).

The mixed phase is favored only when external field $\phi$ is strictly zero. For $\lambda \neq 0$, one min or the other will be favored.
There is a line of 1st order phase transitions along \( \lambda = 0 \), which terminates at the critical point.

Note that a sign of \( \lambda \) is metastable if the external field is weak. \( \lambda \) as a free energy barrier \( \rightarrow \) hysteresis.

**Critical exponents**

Onset of order parameter.

Expand \( q_2(t) \) about its zero:

\[ q_2 \sim \alpha (t - T_c) \quad \alpha > 0 \]

\[ F(e, t) \sim g_0 + \frac{1}{2} \alpha (t - T_c) e^2 + \frac{1}{4} g_4 e^4 \]

(even evaluate \( g_0, g_4 \) at \( t = T_c \))

Minimize: \( \frac{\partial F}{\partial e} = 0 = \alpha (t - T_c) e + g_4 e^3 \)

Solutions: \( e = 0 \) \( e^2 = \frac{\alpha}{g_4} (t_c - t) \)
Minimum is at
\[ \xi = 0 \quad \tau > \tau_c \]
\[ \xi = \pm \left( \frac{\phi_0}{q_4} \right)^{1/2} (\tau_c - \tau)^{1/2} \quad \tau < \tau_c \]

Characteristic mean-field behavior
\[ \beta = \frac{1}{2} \]

Susceptibility
Recall
\[ \chi = \left( \frac{\partial F}{\partial \xi} \right) \tau \]
Define
\[ \chi = \left( \frac{\partial F}{\partial \xi} \right) \tau \quad \chi^{-1} = \left( \frac{\partial^2 F}{\partial \xi^2} \right) \tau \]

\[ \left( \frac{\partial^2 F}{\partial \xi^2} \right) \tau \sim \alpha (\tau - \tau_c) + 3 q_4 \xi^2 + \ldots \]

evaluate at \( \xi_0 \)

So
\[ \chi^{-1} = \alpha (\tau - \tau_c) \quad (\tau > \tau_c) \]
\[ \chi^{-1} = \alpha (\tau - \tau_c) + 3 q_4 \frac{\phi_0}{q_4} (\tau_c - \tau) \]
\[ = 2 \alpha (\tau - \tau_c) \quad (\tau < \tau_c) \]

We find, again, \( \gamma = \gamma' = 1 \), and the factor
2 difference in slope
\[ \chi' \rightarrow \tau \]
Specific Heat

\[ \delta = -\left( \frac{\partial F}{\partial \theta} \right)_V \] is continuous \( \Rightarrow \) no latent heat

Specific heat is

\[ C_V = T \left( \frac{\partial E}{\partial T} \right)_V = -T \left( \frac{\partial F}{\partial T} \right)_V \]

\[ \frac{\partial^2 F}{\partial T^2} = g'''' + g'' E^2 + g' E^4 \]

So

\[ C_V = -T \left[ g''''(T) + g''(T) \right] \frac{1}{2} \frac{\alpha}{\gamma} (T_c - T) \]

\[ T < T_c \]

Thus \( C_V \) is continuous at \( T = T_c \), but its 1st derivative is not continuous.

\[ \delta = 0 \Rightarrow 1 \sim \frac{E}{E_c} \Rightarrow \delta = 3 \]

1st order transition

In the Ising model, most of symmetry breakdown \((\phi = 0)\) was continuous but not analytic. The general theory can also accommodate a discontinuous onset.
E.g. Suppose \( g_4(T_0) < 0 \) and \( g_6(T_0) > 0 \)

where \( g_2(T_0) = 0 \).

**Diagram:**

- **Stable metastable**
- **Metastable stable**

**Remark:**

In some systems, continuous change is not possible. For example, \(-1 \rightarrow 0 \rightarrow 1\) is not a continuous change.

**Order parameter jumps**

- Hysteresis, and latent heat too.

- Different values of \( \xi \):
  \[ \xi = -\frac{2F}{\partial T} \]

**Scaling:**

- Solid-liquid line cannot terminate.

**London theory is simple yet powerful but it is wrong.**

- Specific heat is not continuous—typically, it blows up.
- \( \beta, \gamma \) exponents are wrong.

**Further while there is some tendency for exponents to be universal, this is not completely true experimentally.**

- \( \xi \) can have different values of \( \xi \), \( \beta, \delta \) depending on symmetries of system.

**So the assumption underlying theory—heat \( F \) can be expanded around critical point—is wrong.**
Focus of modern theory of critical phenomena is to understand the singularities of $E(S,T)$ and their origin — great progress made in late 60's and early 70's (Kadanoff, Wilson, ...) 

Key feature that makes mean field approach inapplicable is that, near the critical point, there can be large, long-wavelength fluctuations away from most probable configuration.

\[ \Delta E \]

This happens because $F$ is very flat near $E=0$ — the "restoring force" that opposes an excursion of $E$ becomes weak and ineffective. (Frequency $\rightarrow 0$

Dramatic illustration in laboratory of long-wavelength fluctuations:

- "critical opalescence"

Fluid gets cloudy, becomes regime with $\beta > 0$ or $\beta < 0$ become comparable in size to wavelength of visible light, and so scatter the light.
So as $T \to T_c$, the fluctuations have a characteristic length scale. Fluctuations on all scales are occurring, and we are faced with the problem of understanding a system in which cell scales of length are equally important.

Central idea proposed by Kadanoff, systematically developed by Wilson — idea of "scaling" or "renormalization group." E.g., spins on a lattice: only near neighbors are coupled together by Hamiltonian, but distant spins are in fact strongly correlated because of the long wavelength fluctuations. Describe as a system of interacting spin droplets, with nearby cells of spins coupled together.

Eventually — kings scale — to bigger cells. Eventually — kings scale — to bigger cells are coupled together in some way as the smaller cells (assuming both are small compared to "correlation length," which $\to \infty$ at $T_c$).
\[ F(\vec{r}, \vec{\delta} \vec{c}) = \frac{d}{\vec{c}} F(\vec{r}, \vec{\delta} \vec{c}) \]

\[ \delta \vec{c} = \frac{c - c_c}{c_c} \quad d \text{ : dimensionality (normally 3)} \rightarrow \text{become non-extensive} \]

\[ \vec{c}, \delta \vec{c} \text{ are renormalized} \]

\[ \vec{c} = \frac{\delta \vec{c}}{\delta \vec{c}} \]

\[ \delta \vec{c} = 2^d \delta \vec{c} \]

\[ \vec{c} \text{ powers } \alpha, \beta, \gamma \text{ typically non-integer} \]

If big cells and small cells actually couple the same way, then

\[ F(\vec{r}, \vec{\delta} \vec{c}) = \frac{d}{c} F(\vec{r}, \vec{\delta} \vec{c}) \]

\[ \text{Scaling} \]

\[ \text{Dimensional analysis?} \]

\[ \text{Microscopic distance scale does not drop out} \]

\[ \text{Dimensions dimensions} \]

The powers \( p, q \) are origin of non-analytically behaving of \( F \) and they can be related to critical exponents. Often, they can be calculated.
Remarks:

- Mean field theory will work if fluctuations are not so important. Philosophy of MFT is that each spin interacts with average of other spins — while in fact spin interacts only with nearby spins. Philosophy works better at high dimensionality d.

- There is an "upper critical dimension" at which the Landau prediction of exponents works. For ferromagnet

  \[ d_{upper} = 4 \]

- Conversely, fluctuations are more important for lower d. At sufficiently low dimension, fluctuations destroy long range order at any nonzero temperature. (Ginzburg-Landau)

  \[ d_{lower} = 1 \] (solid magnet)

  \[ d_{lower} = 2 \] (solid magnet)

- e.g. - There is no long range order at any finite T in 1d, hence noLandau 2kink.

- Ladder theory is the starting point of a systematic approximation:

  \[ d = 4 - \epsilon \]

  Calculate exponents as an expansion in \( \epsilon \), and extend to \( \epsilon = 1 \).
Universal}

When the RG-improved Landau theory is used, the concept of universality survives, but in a somewhat restricted sense. Not true that all critical phenomena have the same exponents, but we can identify "universality classes" disjoint phenomena exhibit same exponents (same scaling) — if they have the same underlying symmetries.