Ph 12C Final Exam
Due: Friday, 14 June 2013, 5pm

• This exam is to be taken in one continuous time interval not to exceed 4 hours, beginning when you first open the exam. (You may take one 15 minute break during the exam, which does not count as part of the 4 hours.)

• You may consult the textbook *Thermal Physics* by Kittel and Kroemer, your lecture notes, the online lecture notes, and the problem sets and solutions. If you wish, you may use a calculator or computer for doing calculations. **No other materials or persons are to be consulted.**

• There are four problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.

• The completed exam is to be handed in at the Ph 12 in-box outside 264 Lauritsen. All exams are due at 5pm on Friday June 14. **No late exams will be accepted.**

• Good luck!
1. Condensation of massless bosons — 25 total points

Consider an ideal gas of spinless relativistic bosons in three spatial dimensions, such that a particle’s energy $\epsilon$ is related to its momentum $\vec{p}$ by

$$\epsilon = |\vec{p}| c.$$ 

(a) (10 points) Express the Einstein condensation temperature $\tau_E$ in terms of the particle concentration $n$. You may express your answers in terms of the function $I(\alpha) = \int_0^\infty dx x^\alpha (e^x - 1)^{-1}$.

(b) (5 points) For $\tau \leq \tau_E$, express the fraction $n_0/n$ of particles in the ground orbital in terms of $\tau$ and $\tau_E$.

(c) (10 points) In one spatial dimension, relativistic bosons do not condense at any nonzero temperature. Rather, the number $N_0$ of particles in the ground orbital remains finite as the volume increases to infinity with the concentration $n$ fixed. In this limit, express $N_0$ as a function of $n$ and $\tau$. (Hint: In this case, since $N_0$ is a constant independent of system size, you should not assume that the chemical potential approaches zero in the limit of an infinite system.)

2. Hurricane! — 25 total points

A hurricane is a heat engine, powered by the warm surface of the ocean, which undergoes a four-stroke cycle sketched in the figure.

1. Water evaporates from the ocean surface at temperature $\tau_1$ and pressure $p_1$.
2. Water vapor rises to high altitude, where the temperature is $\tau_2 < \tau_1$ and the pressure is $p_2 < p_1$.
3. Water vapor condenses at temperature $\tau_2$ and pressure $p_2$.
4. Rain falls from high altitude to the ocean surface.

Suppose that a mole of liquid water occupies volume $V_l$ both at sea level and at high altitude and that a mole of water vapor occupies volume $V_g$ both at
sea level and at high altitude. (We assume that any change in volume as the vapor rises or the water falls is negligible.) Suppose that the latent heat per mole required to convert liquid water to water vapor at sea level is $L$.

(a) (5 points) Express the work done on a mole of water during one cycle in terms of $V_l$, $V_g$, $p_1$, $p_2$, $\tau_1$, and $\tau_2$.

(b) (5 points) Assuming that this heat engine achieves the Carnot efficiency, find an expression for $(p_1 - p_2)/(\tau_1 - \tau_2)$. Derive the Clausius-Clapeyron relation by considering the limit $(\tau_1 - \tau_2) \to 0$.

(c) (10 points) If the hurricane has Carnot efficiency, generates power $W$ in Watts, and has area $A$ in m$^2$, at what rate is water evaporating from the ocean surface? Give your answer as the height $h$ in meters of the layer of water that evaporates in one day, expressed in terms of $W$, $A$, $L$, $\tau_1$, and $\tau_2$. (A similar amount of water falls as rain.)

(d) (5 points) A powerful hurricane generates about 200 terawatts ($2 \times 10^{14}$ W) and covers an area of about $10^5$ km$^2$. (In contrast, the world’s power demand is currently only about 15 terawatts.) The ocean surface has temperature $T_1 = 27^\circ$C and the upper atmosphere has temperature $T_2 = -73^\circ$C. The latent heat of vaporization of water is 41 kJ per mole. What is $h$?

3. Generalized Landau Theory — 25 total points

The behavior of macroscopic variables near a second–order phase transition can be characterized by critical exponents. For a ferromagnetic transition, four such exponents, denoted $\alpha$, $\beta$, $\gamma$, $\delta$, may be defined as follows:

$$
C_M \sim (-\epsilon)^{-\alpha}, \quad (H = 0, \epsilon \lesssim 0),
$$

$$
M \sim (-\epsilon)^{\beta}, \quad (H = 0, \epsilon \lesssim 0),
$$

$$
\chi \sim (-\epsilon)^{-\gamma}, \quad (H = 0, \epsilon \lesssim 0),
$$

$$
M \sim H^{1/\delta}, \quad (\epsilon = 0).
$$

Here, $M$ is the magnetization, $H$ is the external applied magnetic field, $C_M = \tau (\partial \sigma / \partial \tau)_M$ is the heat capacity, $\chi = (\partial M / \partial H)_\tau$ is the magnetic susceptibility, and $\epsilon = (\tau - \tau_c)/\tau_c$, where $\tau_c$ is the critical temperature.

In Landau’s theory of second–order phase transitions, the free energy near the critical point is taken to have the form

$$
F \simeq \frac{1}{2} \mu \epsilon M^2 + \frac{1}{4} g M^4,
$$

where $\mu, g > 0$. Under this assumption, critical exponents can be derived; we have seen that

$$
\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3,
$$

in Landau’s theory.

In this problem, we assume a more general form for $F$ near the critical point. For $\epsilon \lesssim 0$, this form is

$$
F \simeq -\frac{1}{b} \mu (-\epsilon)^a M^b + \frac{1}{c} g M^c,
$$

where $\mu, g > 0$. You may assume $a \geq 1$, $c \geq 2$, and $c > b > 0$. 

3
a) (20 points) Use this expression for $F$ to derive $\alpha$, $\beta$, $\gamma$, and $\delta$, expressed in terms of $a$, $b$, and $c$.

b) (5 points) Eliminate $a$, $b$, and $c$ to express $\gamma$ in terms of other exponents.

4. Slightly relativistic Fermi gas — 25 total points

Consider a degenerate ideal gas of mass-$m$ spin-$\frac{1}{2}$ fermions, where the rest energy $mc^2$ is much larger than the Fermi kinetic energy $\epsilon_F$. There are small relativistic corrections to the internal energy $U$ and the pressure $p$ of the gas, which can be calculated by expanding the kinetic energy

$$\epsilon = \left( m^2 c^4 + p^2 c^2 \right)^{1/2} - mc^2$$

in a power series.

a) (15 points) Show that

$$U = \frac{3}{5} N \epsilon_F \left( 1 + A \left( \frac{\epsilon_F}{mc^2} \right) + \cdots \right)$$

and find the numerical constant $A$. (Here $U$ includes only the kinetic energy of the gas, not its rest energy; likewise, $\epsilon_F$ includes only the kinetic energy. The ellipsis indicates further corrections suppressed by additional powers of $\epsilon_F/mc^2$.)

b) (10 points) Show that

$$p = \frac{2}{3} \left( \frac{U}{V} \right) \left( 1 + B \left( \frac{\epsilon_F}{mc^2} \right) + \cdots \right)$$

and find the numerical constant $B$. 