1a) \( P_{l} + \frac{\tau_{l}}{r_{l}} \) \( V = (p_{1} - p_{2})(V_{g} - V_{e}) \), work done by enclosed area

6) Heat added at \( T_{1} \) is \( L \)
\[ W = (p_{1} - p_{2})(V_{g} - V_{e}) = \frac{L}{T_{i}} \frac{T_{1} - T_{e}}{T_{1}} \]
\[ \Rightarrow \frac{p_{1} - p_{2}}{T_{1} - T_{e}} = \frac{L}{T_{i}} \frac{T_{1} - T_{e}}{T_{1}} \]
This agrees with Clausius-Clapeyron in the limit \( T_{e} \rightarrow T_{1} \).

\( \left( \frac{dp}{dT} \right)_{coex} = \frac{L}{T_{i}} \frac{V AV}{C_{p}} \) It is a derivative of \( C_{p} \) since cycle is reversible.

c) Power \( W = 4 \eta \frac{L}{sec} \times \left( \frac{mole}{sec} \right) \), \( \eta = \frac{T_{1} - T_{e}}{T_{1}} \)
\[ \frac{Vol}{sec} = \frac{mole}{sec} \times \frac{m^3}{mole} = \frac{mole}{sec} \times (18 \times 10^{-6}) \]
\( = 18 \) molecular wt. of \( H_{2}O \)
\[ \frac{height}{sec} = \frac{Vol}{area} \frac{sec}{sec} = \frac{1}{A} (18 \times 10^{-6}) \frac{W}{\eta L} \]
\[ \frac{height}{day} = (24 \times 3600) \times (18 \times 10^{-6}) \frac{W}{AL\eta} \]
\[ = 1.56 \frac{W}{AL\eta} \]

d) \( \eta = \frac{T_{1} - T_{e}}{T_{1}} = \frac{300 - 200}{300} = \frac{1}{3} \)
\[ \frac{height}{day} = (1.56)(3) \frac{2 \times 10^{14} \text{Wh/day}}{(10^4 \text{m}^2)(41 \times 10^{3} \text{J/m}^2\text{C}^2)} = 2.2 \frac{m}{\text{day}} \]
9 inches of rain per day — sounds about right
Physics 12c, Problem Set 7 Solutions

May 2016

[2] Scaling hypothesis from Landau theory

From
\[ G(\epsilon, \lambda) = [F(\epsilon, \xi) - \lambda \xi] \] stat wrt \( \xi \)
and
\[ F(\epsilon, \xi) = \frac{1}{2} A \epsilon \xi^2 + \frac{1}{4} B \xi^4, \]
we infer that
\[ G(\Omega p \epsilon, \Omega q \lambda) = \left[ \frac{1}{2} A \epsilon \xi^2 + \frac{1}{4} B \xi^4 - \lambda \Omega \xi \right] \] stat wrt \( \xi \).

Now we want to show that the right-hand side becomes \( \Omega G(\epsilon, \lambda) \) for an appropriate choice of \( p \) and \( q \). In fact if we choose \( (p-1)/2 = -1/4 \) and \( q-1 = -1/4 \), then the quantity inside the square brackets becomes a function of \( \Omega^{-1/4} \xi \), and we have
\[ G(\Omega p \epsilon, \Omega q \lambda) = \left[ \frac{1}{2} A \epsilon \left( \Omega^{-1/4} \xi \right)^2 + \frac{1}{4} B \left( \Omega^{-1/4} \xi \right)^4 - \lambda \left( \Omega^{-1/4} \xi \right) \right] \] stat wrt \( \xi \).

But rescaling \( \xi \) by \( \Omega^{-1/4} \) does not change the value of the quantity in square brackets at its stationary point, and so we find
\[ G(\Omega p \epsilon, \Omega q \lambda) = \Omega G(\epsilon, \lambda) \]
when we choose
\[ p = 1/2, \quad q = 3/4. \]

[3] Critical exponents from the scaling hypothesis

(a) The order parameter is given by \( \xi = -\left( \frac{\partial G}{\partial \lambda} \right)_\tau \). Differentiating the scaling hypothesis with respect to \( \lambda \) gives:
\[ \Omega^q \xi(\Omega^p \epsilon, \Omega^q \lambda) = \Omega \xi(\epsilon, \lambda) \] (S1)

We set \( \lambda = 0 \), and take the limit \( \epsilon \to 0^- \) (i.e. approach \( \tau_C \) from below) while holding \( \Omega^p \epsilon \) fixed. This means that \( \Omega \propto |\epsilon|^{-1/p} = (-\epsilon)^{-1/p} \) since \( \epsilon < 0 \) in this case\(^1\). Furthermore, for \( \Omega^p \epsilon \) fixed, \( \xi(\Omega^p \epsilon, 0) \) is just a constant. Therefore,
\[ \xi \sim \Omega^{q-1} \sim (-\epsilon)^{\frac{1-q}{p}} \quad \Rightarrow \quad \beta = \frac{1-q}{p}. \]

\(^1\) We usually want to study the scaling behavior while on the same side of \( \tau_C \), hence we can take \( \Omega \) as a positive quantity.
(b) Set $\epsilon = 0$ in eq. (S1), and take the limit $\lambda \to 0$, while holding $\Omega^\xi$ fixed, i.e. $\Omega \propto \lambda^{1/q}$. Therefore, we get

$$\xi \sim \Omega^{q-1} \sim \lambda^{-\frac{q-1}{q}}$$

$$\Rightarrow \quad \lambda \sim \xi^{\frac{q}{q-1}} \quad \Rightarrow \quad \delta = \frac{q}{1-q}.$$  

(c) Recall that $\sigma = -\left(\frac{\partial G}{\partial \tau}\right)_\lambda$, so the heat capacity is given by $C_\lambda = -\tau \left(\frac{\partial^2 G}{\partial \tau^2}\right)_\lambda$.

Differentiating the scaling hypothesis twice with respect to $\tau$, we get

$$\Omega^{2p} C_\lambda(\Omega^p \epsilon, \Omega^q \lambda) = \Omega^p C_\lambda(\epsilon, \lambda).$$

Set $\lambda = 0$, and take the limit $\epsilon \to 0$ while holding $\Omega^p \epsilon$ fixed, i.e. $\Omega \propto |\epsilon|^{-1/p}$. Then,

$$C_\lambda \sim \Omega^{2p-1} \sim |\epsilon|^{-\frac{2p-1}{p}} \Rightarrow \alpha = 2 - \frac{1}{p}.$$  

(d) For $p = 1/2$ and $q = 3/4$, we find $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$ as expected.

(e) Using the expressions for $\beta$ and $\delta$ from problem 2, we can write:

$$\delta = \frac{q}{1-q} \quad \Rightarrow \quad q = \frac{\delta}{1+\delta},$$

$$\beta = \frac{1-q}{p} = 2 - \alpha - \frac{q}{p} \quad \Rightarrow \quad \frac{1}{p} = \beta(1+\delta).$$

Therefore, $\alpha = 2 - \frac{1}{p} = 2 - \beta(1+\delta)$, which is known as the Griffiths relation.

(f) Using the expressions for $\alpha$ and $\beta$ from problem 2, we can write:

$$\alpha = 2 - \frac{1}{p} \quad \Rightarrow \quad \frac{1}{p} = 2 - \alpha,$$

$$\beta = \frac{1-q}{p} = 2 - \alpha - \frac{q}{p} \quad \Rightarrow \quad \frac{q}{p} = 2 - \alpha - \beta.$$  

Therefore, $\gamma = \frac{2q}{p} - \frac{1}{p} = 2(2 - \alpha - \beta) - (2 - \alpha) = 2 - \alpha - 2\beta$, which is known as the Rushbrooke relation.

[4] Equation of state from the scaling hypothesis

(a) Differentiating both sides of the scaling hypothesis

$$G(\epsilon, \lambda) = \Omega^{-1} G(\Omega^p \epsilon, \Omega^q \lambda),$$

we find

$$\xi(\epsilon, \lambda) = -\left(\frac{\partial G}{\partial \lambda}\right)_\tau = \Omega^{q-1} \xi(\Omega^p \epsilon, \Omega^q \lambda).$$

Now choose $\Omega$ so that $\Omega^p = \epsilon^{-1}$, or $\Omega = \epsilon^{-1/p}$, and we have

$$\xi(\epsilon, \lambda) = \epsilon^{(1-q)/p} \xi \left(1, \frac{\epsilon^{-q/p} \lambda}{\epsilon^{(1-q)/p}}\right) \Rightarrow \xi(\epsilon, \lambda) = \xi \left(1, \frac{\lambda}{\epsilon^{(1-q)/p}}\right) = f \left(\frac{\lambda}{\epsilon^{(1-q)/p}}\right).$$

Therefore,

$$a = \frac{1-q}{p} = \beta, \quad b = \frac{q}{p} = \beta \delta.$$  

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(b) Differentiating we find

\[ \lambda = \frac{\partial}{\partial \xi} F(\epsilon, \xi) = A\epsilon \xi + B\xi^3, \]

and therefore

\[ \lambda \epsilon^{-b} = A\epsilon^{1-b} \xi + B\epsilon^{-b} \xi^3 = A (\epsilon^{1-b} \xi) + B \left( \epsilon^{-b/3} \xi \right)^3; \]

This has the form \( h(\xi/\epsilon^a) \) if \( a = b - 1 = b/3 \), which has the solution \( b = 3/2 \) and \( a = 1/2 \). The function \( h \) is

\[ h(x) = Ax + Bx^3. \]

To check: in Landau theory, where \( p = 1/2 \) and \( q = 3/4 \), the result from (a) becomes

\[ a = \frac{1 - q}{p} = \frac{1/4}{1/2} = 1/2, \quad b = \frac{q}{p} = \frac{3/4}{1/2} = 3/2. \]