Ph 12c

Homework Assignment No. 3
Due: 5pm, Thursday, 25 April 2013

1. Surface temperature of Venus

a) Suppose that Venus is a black body with uniform surface temperature T_V. (Ignore that the daytime side is actually warmer than the nighttime side of the planet.) In equilibrium it radiates energy into space at the same rate that it absorbs energy from sunlight. Express T_V in terms of the temperature $T_\odot$ of the sun, the radius $R_\odot$ of the sun, and the distance D from the sun to Venus.

b) In part (*a*) we ignored the atmosphere of Venus, which has two important effects. The thick clouds of Venus reflect a fraction $1-\eta$ of the sunlight, allowing a fraction η to pass through and reach the surface. Furthermore, the clouds are opaque to the infrared emission from the surface (the "greenhouse effect"). Model the greenhouse effect as a spherical shell below the clouds and above the surface (with the same radius as the planet), which is transparent to sunlight but absorbs all of the infrared radiation incident on it. Find a revised expression for T_V that takes into account the reflectivity of the clouds and the infrared absorption by the atmosphere. Note that the shell is heated by the thermal emission from the surface below, and re-radiates in two directions, down toward the surface and out into space. Assume that the outward radiation from the shell passes through the clouds without any absorption.

c) In part (*b*) we assumed that the atmosphere of Venus has a uniform temperature, but actually the lower atmosphere is much warmer than the upper atmosphere. Model this effect by assuming that Venus is surrounded by N shells, each perfectly absorbing in the infrared, which are thermally isolated from one another except for the exchange of thermal radiation between them. Each shell radiates upward and downward with the same temperature. Revise your express for T_V from part (*b*) taking into account the N shells. You may find it convenient to consider the outermost shell first: in equilibrium, how much power is radiated outward from this shell? How is the power radiated outward from each shell related to the power radiated outward from the next shell above it?

d) Using $R_\odot = 7.0 \times 10^5$ km $T_\odot = 5800^\circ$ K, $D = 1.1 \times 10^8$ km, $\eta = .23$, estimate the number of shells N needed to account for the observed surface temperature $T_V \approx 740^\circ$ K. If you find $N \gg 1$, that means the atmosphere of Venus is quite thick compared to the distance infrared radiation can travel before being absorbed. (We say that the atmosphere has high "optical depth" in the infrared.)

2. Debye theory of capillary waves

Ripples of small amplitude and short wavelength on the surface of a liquid are called "capillary waves". They are governed by surface tension, and obey a dispersion relation of the form

$$\omega^2 = Ck^3.$$

Here ω is the circular frequency, k is the wave number, and C is a constant. There is a single polarization for each wave vector.

Derive the analog of the Debye formula for the energy per unit surface area stored in the thermal surface vibrations, in the limit of low temperature τ. Express your answer in the form

$$U/A = \alpha\ C^\beta\ \hbar^\gamma\ \tau^\delta,$$

where U is energy, A is surface area, and α, β, γ, δ are numbers you are to find. As a check, verify the dimensional consistency of your answer. To evaluate α, you might want to use the integral

$$I = \int_0^\infty \frac{x^{4/3}\ dx}{e^x - 1} \simeq 1.685\ .$$

3. Quantum noise and thermal noise in a harmonic oscillator

You may recall from Ph 12b that the position x and the momentum p of a harmonic oscillator with circular frequency ω and mass m may be expressed as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \quad p = -i\sqrt{\frac{\hbar m\omega}{2}}\left(a - a^\dagger\right),$$

where $[a, a^\dagger] = 1$. Hence $[x, p] = i\hbar$, and the Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).$$

It is convenient to describe the dynamics of the oscillator using the *Heisenberg picture* operator

$$x(t) = e^{iHt/\hbar}\ x\ e^{-iHt/\hbar} = \sqrt{\frac{\hbar}{2m\omega}}\left(a\ e^{-i\omega t} + a^\dagger\ e^{i\omega t}\right).$$

The expectation value of $x(t)x(0)$ is said to be the *time correlation function* for the oscillator's position; it quantifies how the position at time t is correlated with the position at time 0, and hence characterizes the fluctuations in position.

a) Let $|n\rangle$ denote the nth excited state of the oscillator, with energy $\hbar\omega(n + \frac{1}{2})$. Evaluate the expectation value

$$\langle n|\ x(t)x(0)\ |n\rangle.$$

In the thermal ensemble at temperature τ, the probability that the oscillator's state is $|n\rangle$ is proportional to the Boltzmann factor $\exp(-n\hbar\omega/\tau)$. The expectation value $\Delta_\tau(t)$ of $x(t)x(0)$ in this thermal ensemble has the form

$$\Delta_\tau(t) \equiv \langle x(t)x(0)\rangle_\tau = P_\tau(\omega)e^{-i\omega t} + N_\tau(\omega)e^{i\omega t},$$

where $P_\tau(\omega)$ and $N_\tau(\omega)$ are functions of ω. We say that $P_\tau(\omega)e^{-i\omega t}$ is the "positive frequency" part of the correlation function $\Delta_\tau(t)$ and that $N_\tau(\omega)e^{i\omega t}$ is the "negative frequency" part.

b) Find $P_\tau(\omega)$ and $N_\tau(\omega)$; show that

$$\frac{N_\tau(\omega)}{P_\tau(\omega)} = e^{-\hbar\omega/\tau}.$$

c) Evaluate $\Delta_\tau(t)$ in the limit $\tau \to 0$ and in the limit $\tau \to \infty$. Check that in the high-temperature limit the correlation function does not depend on $\hbar$ — it describes "classical" thermal fluctuations of the oscillator.

The *Kubo-Martin-Schwinger (KMS) condition* is a general property of fluctuations in thermal equilibrium at temperature τ, saying that fluctuations at negative frequency $-\omega$ are suppressed relative to fluctuations at positive frequency ω by the Boltzmann factor $e^{-\hbar\omega/\tau}$; in (b) you have verified the KMS condition for the special case of the position of a harmonic oscillator.

4. Lifetime of a black hole

A black hole with mass M (and hence energy Mc^2) has surface area $A = 4\pi R^2$, where $R = 2GM/c^2$ is its "Schwarzschild radius." Its entropy is

$$\sigma = \frac{A}{4L_{\mathrm{Pl}}^2},$$

where

$$L_{\mathrm{Pl}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35} m,$$

is the "Planck length" (G is Newton's gravitational constant).

a) Find the temperature τ of a black hole, expressed in terms of M, G, $\hbar$, and c.

b) Black holes evaporate. Assuming the black hole radiates like a black body with temperature τ and surface area A, show that its mass $M(t)$ decreases as a function of time according to

$$M^2 \frac{dM}{dt} = -B$$

where B is a constant. Express B in terms of G, $\hbar$, and c.

c) By solving this differential equation, find the time t_M for a black hole to evaporate completely if its initial mass is M. Express t_M in terms of B and M.

d) What is the lifetime of a solar mass black hole? (Don't be surprised if it's a long time.)