Ph 12c

Homework Assignment No. 2
Due: 5pm, Thursday, 18 April 2013

Do Problems 4 and 11 in Chapter 3 of Kittel and Kroemer, plus these additional problems:

1. Model of a large reservoir

a) Consider a system $\mathcal{S}$ divided into two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ in thermal contact, sharing total energy E. If $\mathcal{S}_1$ has energy E_1 and $\mathcal{S}_2$ has energy $E_2 = E - E_1$, the total entropy of $\mathcal{S}$ is

$$S_{\text{total}} = S_1(E_1) + S_2(E_2)$$

where S_1 is the entropy of $\mathcal{S}_1$ and S_2 is the entropy of $\mathcal{S}_2$. Show that if E_1 is chosen to maximize S_{total} ("the most probable configuration") with the total energy E fixed, then the two subsystems have the same temperature: $\tau_1 = \tau_2$. (To verify that this configuration is really a maximum rather than a minimum, check the sign of the second derivative of S_{total} with respect to E_1, assuming the "heat capacity" $C_i = dE_i/d\tau_i$ is positive for both subsystems.)

b) Now suppose $\mathcal{S}$ is divided into N subsystems $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N$ in thermal contact; with total entropy

$$S_{\text{total}} = \sum_{i=1}^{N} S_i(E_i)$$

where S_i, E_i are the entropy and energy of $\mathcal{S}_i$. Using mathematical induction and part (a), show that if the total energy $E = E_1 + E_2 + \cdots + E_N$ is fixed, then the total entropy S_{total} is maximized when all N systems have the same temperature.

c) Now consider a large reservoir consisting of N identical subsystems, all in thermal contact with one another and each with the same entropy function $S(E)$. It follows from part (b) that in the most probable configuration all subsystems have the same temperature and all therefore have the same energy as well; hence the total entropy is

$$S_{\text{total}}(E) = NS(E/N),$$

where E is the total energy. If the total energy increases from E to $E+\Delta E$, find the corresponding change ΔS_{total} in the total entropy, expanded in a power series to quadratic order in ΔE. Express your answer in terms of the reservoir's temperature τ and the heat capacity $C = dE/d\tau$ of each *subsystem*. Argue that it is reasonable to neglect the term of order $(\Delta E)^2$ when the number of subsystems is $N \gg 1$.

2. Atoms and photons

$N \gg 1$ identical atoms are in equilibrium with radiation in a cavity at temperature τ. Each atom has two energy eigenstates: the ground state $|g\rangle$ with energy E_g, and the excited state $|e\rangle$ with energy E_e, where $E_e - E_g = \hbar\omega$. Occasionally an atom absorbs a photon and makes a transition from the ground state to the excited state, or emits a photon and makes a transition from the excited state to the ground state. Let $N(g)$ denote the number of atoms that occupy the ground state, let $N(e) = N - N(g)$ denote the number of atoms that occupy the excited state, let $\Gamma(g \to e)$ denote the rate (probability per unit time) for an atom in the ground state to absorb a photon, and let $\Gamma(e \to g)$ denote the rate for an atom in the excited state to emit a photon.

(a) Show that, because $N(g)$ remains constant in equilibrium, it follows that

$$N(g)\Gamma(g \to e) = N(e)\Gamma(e \to g). \tag{1}$$

(b) We observe that, for atoms in equilibrium with the thermal radiation, the rate $\Gamma(g \to e)$ is half as large as the rate $\Gamma(e \to g)$. What is the temperature τ?

3. Anisotropic well

The Hamiltonian for a particle of mass m in an anisotropic potential well is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{m}{2}\left(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2\right). \tag{2}$$

Since H is the sum of three one-dimensional harmonic oscillator Hamiltonians with circular frequencies $\omega_1, \omega_2, \omega_3$, the energy eigenvalues are

$$E(n_1, n_2, n_3) = \hbar\omega_1 n_1 + \hbar\omega_2 n_2 + \hbar\omega_3 n_3 \tag{3}$$

(ignoring the zero point energy), where n_1, n_2, n_3 are non-negative integers.

(a) Find the partition function Z_1 for a single particle in the potential well at temperature τ.

(b) Now suppose that N distinguishable non-interacting particles are in the potential well. Express the partition function Z_N in terms of the single-particle partition function Z_1.

(c) Compute the average energy $U(\tau, N)$.

(d) Find the heat capacity $C = (\partial U/\partial \tau)_N$ in the high-temperature limit, $\tau \gg \hbar\omega_1, \hbar\omega_2, \hbar\omega_3$.