Ph 12b Quantum Physics
4 February 2010
Uncertainty Principle

Note Title 2/7/2010

Schwarz inequality: $|\langle\varphi|\psi\rangle| \le \|\psi\|\,\|\varphi\|$

Proof: use completeness

$$\hat{I} = \sum_i |e_i\rangle\langle e_i| \Rightarrow |\psi\rangle = \sum_i |e_i\rangle\langle e_i|\psi\rangle$$

$$\Rightarrow \|\psi\|^2 = \langle\psi|\psi\rangle = \sum_i |\langle e_i|\psi\rangle|^2 \geq |\langle e_1|\psi\rangle|^2.$$

If $|\varphi\rangle = 0$ then Schwarz inequality is trivial.

If $|\varphi\rangle \neq 0$, choose $|e_1\rangle = \dfrac{|\varphi\rangle}{\|\varphi\|}$

$$\Rightarrow \|\psi\|^2 \geq \frac{|\langle\varphi|\psi\rangle|^2}{\|\varphi\|^2} \Rightarrow \|\psi\|\,\|\varphi\| \geq |\langle\varphi|\psi\rangle|$$

Variance of observable $\hat{A}$ in state $|\psi\rangle$ is

$$(\Delta A)^2 = \langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle = \langle\psi|(\hat{A}-\langle\hat{A}\rangle)^2|\psi\rangle = \|(\hat{A}-\langle\hat{A}\rangle)|\psi\rangle\|^2$$

($\hat{A}$ is Hermitian).

Schwarz inequality $\Rightarrow$

$$(\Delta A)^2(\Delta B)^2 = \|(\hat{A}-\langle\hat{A}\rangle)|\psi\rangle\|^2 \;\|(\hat{B}-\langle\hat{B}\rangle)|\psi\rangle\|^2$$

$$\geq |\langle\psi|(\hat{A}-\langle\hat{A}\rangle)(\hat{B}-\langle\hat{B}\rangle)|\psi\rangle|^2$$

Let $\hat{C} = \hat{A} - \langle\hat{A}\rangle$, $\hat{D} = \hat{B} - \langle\hat{B}\rangle \Rightarrow$

$$\hat{C}\hat{D} = \hat{E} + i\hat{F}, \text{ where}$$

$$\hat{E} = \tfrac{1}{2}(\hat{C}\hat{D} + \hat{D}\hat{C}) \qquad \hat{F} = \tfrac{-i}{2}(\hat{C}\hat{D} - \hat{D}\hat{C})$$

$\hat{C}$ and $\hat{D}$ are Hermitian $\Rightarrow$ $\hat{E}$ and $\hat{F}$ are Hermitian,

because $(\hat{C}\hat{D})^\dagger = \hat{D}^\dagger\hat{C}^\dagger = \hat{D}\hat{C}$

Therefore $\langle\psi|\hat{E}|\psi\rangle$ and $\langle\psi|\hat{F}|\psi\rangle$ are both real numbers

$$\Rightarrow |\langle\psi|\hat{C}\hat{D}|\psi\rangle|^2 = \langle\psi|\hat{E}|\psi\rangle^2 + \langle\psi|\hat{F}|\psi\rangle^2 \geq$$
$$\geq \langle\psi|\hat{F}|\psi\rangle^2 = \frac{1}{4}|\langle\psi|[\hat{C},\hat{D}]|\psi\rangle|^2$$

$$\hat{C} = \hat{A} - \langle\hat{A}\rangle, \quad \hat{D} = \hat{B} - \langle\hat{B}\rangle \Longrightarrow [\hat{C},\hat{D}] = [\hat{A},\hat{B}]$$

Finally, we have derived (taking positive square roots):

Uncertainty Principle:

$$\boxed{\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|}$$

For position and momentum:

$$[\hat{x},\hat{p}] = i\hbar\hat{I} \Longrightarrow \boxed{\Delta x \Delta p \geq \frac{1}{2}\hbar}$$

or in terms of wave number $\hat{k}$, where $\hat{p} = \hbar\hat{k}$:

$$\Delta x \Delta k \geq \frac{1}{2}.$$

Under what conditions does this uncertainty inequality become an inequality?

Schwarz inequality becomes an equality for $|\langle\varphi|\psi\rangle| = ||\varphi||\,||\psi||$

or $\quad |\varphi\rangle = \lambda|\psi\rangle \quad$ ($|\varphi\rangle$ and $|\psi\rangle$ are parallel vectors).

This condition becomes $\quad \hat{C}|\psi\rangle = \lambda\hat{D}|\psi\rangle \quad (\lambda \in \mathbb{C})$

in the derivation above.

Also $\quad \langle\psi|\hat{E}|\psi\rangle^2 + \langle\psi|\hat{F}|\psi\rangle^2 \geq \langle\psi|\hat{F}|\psi\rangle^2$

becomes an equality if $\; 0 = \langle\psi|\hat{E}|\psi\rangle = \frac{1}{2}\langle\psi|\hat{C}\hat{D}+\hat{D}\hat{C}|\psi\rangle$

Substituting $\hat{C}|\psi\rangle = \lambda\hat{D}|\psi\rangle$ into this equation yields

$$0 = \frac{1}{2}(\lambda+\lambda^*)\langle\psi|\hat{D}^2|\psi\rangle = \mathrm{Re}(\lambda)\,||\hat{D}|\psi\rangle||^2$$

Therefore, either $\mathrm{Re}(\lambda) = 0$ or $\hat{D}|\psi\rangle = 0$

But -- $\hat{D}|\psi\rangle = 0$ and $\hat{C}|\psi\rangle = \lambda\hat{D}|\psi\rangle \Rightarrow \hat{C}|\psi\rangle = 0$

In that case $\hat{A}|\psi\rangle = \langle\hat{A}\rangle|\psi\rangle$, $\hat{B}|\psi\rangle = \langle\hat{B}\rangle|\psi\rangle$;

thus $|\psi\rangle$ is a simultaneous eigenstate of $\hat{A}$ and $\hat{B}$.

otherwise λ is imaginary $\Rightarrow$ we write $\lambda = -i\gamma$
where γ is real

$\Rightarrow \hat{C}|\psi\rangle = -i\gamma\hat{D}|\psi\rangle \Rightarrow (\hat{C} + i\gamma\hat{D})|\psi\rangle = 0$

$\Rightarrow (\hat{A} + i\gamma\hat{B})|\psi\rangle = \langle\hat{A} + i\gamma\hat{B}\rangle|\psi\rangle$.

Thus $|\psi\rangle$ is an eigenvector of the (non-Hermitian) operator $\hat{A} + i\gamma\hat{B}$.

Of course, this statement is also true if $|\psi\rangle$ is a simultaneous eigenstate of $\hat{A}$ and $\hat{B}$. We conclude that:

Condition for equality in uncertainty relation:

$\Delta A \Delta B = \frac{1}{2}\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle$ if and only if

$|\psi\rangle$ is an eigenstate of $\hat{A} + i\gamma\hat{B}$ for some real number γ.