8. The Finite Well and The Periodic Potential

Read §8.2 - §8.4

Problems due
7.62 co-dec 
7.63 + decay 8.2 Finite Well
7.87 free particle action 8.7 Semiinfinite well
7.92 waveparticle in well 8.12 periodic S function

Finite Well

What are bound states in a rectangular potential well of finite depth?

Let the width be $2a$ and the depth be $V_W$.

Because of the symmetry of the potential, we know that the eigenfunctions are either even or odd.

Even soln: inside $\cos Kx$ \[ K^2 = \frac{2m}{\hbar^2} (V_W - 1E) \] outside $e^{-Kx}$

Odd soln: inside $\sin Kx$ \[ \frac{K \cos Ka}{\sin Ka} = \frac{-K e^{-Ka}}{e^{-Ka}} \Rightarrow K \tan Ka = K \]

These equations determine the eigenenergies.
Dimensionless Variables:

Let \( \xi = Ka \Rightarrow \xi^2 + \eta^2 = \frac{2m}{\hbar^2} \nu 1a^2 = \epsilon^2 \)

\( \eta = Ka \)

("coupling constant")

Even solutions \( \xi \tan \xi = \eta \)

Odd solutions \( -\xi \cot \xi = \eta \)

We can imagine solving these equations graphically.

Intersection of the quarter circle with the curves

\( \eta = \xi \tan \xi, -\xi \cot \xi \)

are where bound states occur.

We see that for \( \rho \) small \( (\rho < \frac{\pi}{2}) \) there is only one bound state. And each time \( \rho \) increases by \( \frac{\pi}{2} \), another bound state "enters" the potential.

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Two interesting limits:

1. If infinite well, let \( \rho \to \infty \), and look at the most deeply bound states, i.e., the smallest values of \( \xi = Ka \)

As radius \( \to \infty \), the circle crosses that curve at \( \xi = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{etc} \)

or \( Ka = \frac{n\pi}{2}, \text{integer} \Rightarrow \rho = \frac{\pi}{2a}, \text{integer} \)

- i.e., integer \# of half wavelengths \( K \)s in \( K \) well (as seen previously)

\( E-V = \frac{n^2\pi^2}{2a^2} \) energy measured from bottom of well.
**The δ-function**  

Evaluating the limit \( q \to 0 \) with \( 1/VlA \) fixed.

In this limit \( q \to 0 \), so we solve

\[
\begin{align*}
\delta^2 + \eta^2 &= \epsilon^2 \\
\eta &= \delta \tan \delta x - \delta^2 \\
K &= \frac{2m}{\epsilon^2} \sqrt{1/VlA}
\end{align*}
\]

\[
\Rightarrow K = \frac{m}{\epsilon^2} \sqrt{1/Vl(2a)}
\]

We can easily verify this result by solving the δ-function problem directly:

Let \( V(x) = -\frac{\epsilon^2 A}{m} S(x) \) - attractive δ-function

i.e. Schrödinger equation is

\[
\left[ -\frac{\epsilon^2}{2m} \frac{d^2}{dx^2} - \frac{\epsilon^2 A}{m} S(x) \right] \phi(x) = E \phi(x)
\]

Integrate both sides

\[
\int_{-\infty}^{\infty} \phi^* dx = -\epsilon
\]

\[
\Rightarrow -\frac{\epsilon^2}{2m} \left[ \phi'(-\epsilon) - \phi'(-\epsilon) + 2A \phi(0) \right] = 0
\]

\[
\Rightarrow \frac{\phi'}{\epsilon} \bigg|_{-\epsilon} - \frac{\phi'}{\epsilon} \bigg|_{-\epsilon} = -2A
\]

i.e. \( -2A \) make discontinuity in the log derivative at the origin.

The bound state solution is \( e^{-Klx} \) - with discontinuity

\(-2K\) in log derivative \( \Rightarrow K = \Delta \)
and so \( K = A \)

And in the function limit of the finite well
\[
\frac{m}{\hbar^2} 4U / (2a) = \Delta = K \text{ was found above}
\]
\[
\Rightarrow K^2 = \frac{2m}{\hbar^2} 4E / 1 = \Delta^2
\]
\[
\Rightarrow E = -\frac{\hbar^2}{2m} \Delta^2
\]

Reflection and Transmission from 5 function barrier

\[
\begin{array}{c|c}
\Delta < 0 & \text{For } \Delta < 0, \text{ we have a repulsive 5 function. This is also a limit of a rectangular barrier.}
\end{array}
\]

But let's solve the problem from the beginning...

\[
\begin{array}{c|c}
\text{I} & \text{II} \text{ In fact, it is simple enough that we can solve the problem in the case where the wave in region II is not purely outgoing}
\end{array}
\]

\[
\begin{align*}
\Phi_{\text{I}} &= A e^{ikx} + B e^{-ikx} \\
\Phi_{\text{II}} &= C e^{ikx} + D e^{-ikx}
\end{align*}
\]

\[
\begin{align*}
&\text{match at origin:} \\
&\Rightarrow A + B = C + D \\
&iK(A - B) = iK(C - D) + 2\Delta(A + B)
\end{align*}
\]

Eliminate \( D \):
\[
\Rightarrow iK2C = iK2A - 2\Delta(A + B)
\]
\[
\Rightarrow C = (1 + i\frac{\Delta}{K})A + i\frac{\Delta}{K}B
\]

and \( D = A + B - C = -i\frac{\Delta}{K}A + (1 - i\frac{\Delta}{K})B \)
Write this as a matrix:

\[
\begin{pmatrix}
C \\
D
\end{pmatrix} = \begin{pmatrix}
1+i\alpha & i\alpha \\
-i\alpha & 1-i\alpha
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix}
\]

\[\alpha = \frac{A}{K}\]

(Note: we can check current conservation:

\[|C|^2 - |D|^2 = (A)^2 - |B|^2\]

This matrix has determinant = 1, and is easy to invert:

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
1-i\alpha & -i\alpha \\
i\alpha & 1+i\alpha
\end{pmatrix} \begin{pmatrix}
C \\
D
\end{pmatrix}
\]

In the case where there is not incoming wave in region II, we have \(D=0\) \(\Rightarrow\)

\[
\frac{A}{C} = 1-i\alpha \quad \frac{B}{C} = i\alpha
\]

\[
T' = \left|\frac{C}{A}\right|^2 = \left|\frac{1}{1-i\alpha}\right|^2 = \frac{1}{1+\alpha^2}
\]

\[
R = \left|\frac{B}{A}\right|^2 = \left|\frac{i\alpha}{1-i\alpha}\right|^2 = \frac{\alpha^2}{1+\alpha^2} \Rightarrow R+T' = 1
\]

Let's check that this agrees with limiting behavior for the case of the rectangular barrier:

Consider \(2a \rightarrow 0\) \(\frac{m}{E^2} \cdot V(2a) = -4\) fixed

We had found \(\frac{1}{T'} = 1 + \frac{V^2}{4E(V-E)} \sinh^2(K_z 2a)\)

\[K_z^2 = \frac{2m}{h^2} \cdot (V-E)\]
Note: the transmission amplitude
\[ \frac{c}{d} = \frac{1}{1-ix} \]
has a pole singularity at \( x = \frac{d}{k} = -i \) or \( k = i\Delta \).

Why? Recalling that
\[ \psi_1 = Ae^{ikx} + Be^{-ikx} \rightarrow Ae^{-\Delta x} + Be^{\Delta x} \]
\[ \psi_2 = Ce^{-\Delta x} \]

For \( \Delta > 0 \),
we recognize this as the bound state solution, with \( B = C \) and \( A = 0 \). That is \( \psi \) blows up because we have a decaying exponential in region II with no corresponding increasing exponential in region I!

A pole in the transmission amplitude, and at \( k_{2m} = -1E_0 \), is a general consequence of a bound state with binding energy \( 1E_0 \).
As \(2a \to 0\),

\[
\sinh^2(k_z 2a) \approx k_z^2 (2a)^2 = \frac{2m}{\hbar^2} (V-E)(2a)^2
\]

\[
\Rightarrow \frac{1}{n^2} = 1 + \frac{V}{4E} \frac{2m}{\hbar^2} (2a)^2
\]

\[
= 1 + \frac{1}{2E} \frac{\hbar^2}{m} \Delta^2 = 1 + \frac{a^2}{\hbar^2}
\]

(Which does agree with \(\frac{1}{n^2} = 1 + a^2\).)

**Periodic Potential**

One of the great triumphs of quantum mechanics is the theory of electrons in metals, semiconductors and insulators... How do electrons manage to move through a crystal? The atoms are packed tight... there isn't much room to squeeze through.

Yet somehow, in a conductor, the electrons manage to drift through... how do they do it?

The electrons don't actually squeeze through the atoms; they tunnel through. How can they tunnel through a macroscopic crystal?

For the electrons to perform this miracle, it is crucially important that the atoms...