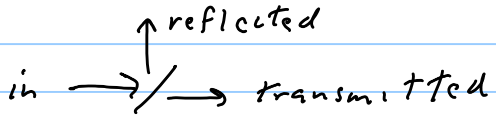
Ph 12b -- Quantum Physics
21 January 2010
The density operator

Note Title 1/19/2010

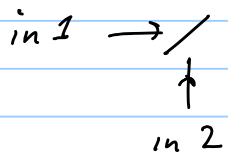
## Quantum bomb testing

Let's consider another way to combine unitary evolution with measurement, which illustrates how the quantum postulates can lead to surprising conclusions.

First consider a photon beam splitter, a partially silvered mirror which reflects a photon with probability $\frac{1}{2}$ and transmits it with probability $\frac{1}{2}$.

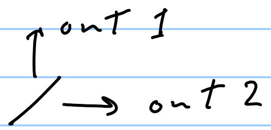



We can describe how the beam splitter acts on a single incident photon using a unitary $2\times 2$ matrix. The photon can enter the beam splitter via either one of two input "ports" which we label in 1 and in 2. There are thus two mutually orthogonal input states,




$$|\text{in } 1\rangle \quad \text{and} \quad |\text{in } 2\rangle$$

For either of these input states, the photon is reflected with probability $\frac{1}{2}$, but the beam splitter preserves the orthogonality of the states. Denoting the two possible states exiting the beam splitter $|\text{out } 1\rangle$ and $|\text{out } 2\rangle$




we have
$$|\text{in } 1\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|\text{out } 1\rangle + |\text{out } 2\rangle\right)$$
$$|\text{in } 2\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(-|\text{out } 1\rangle + |\text{out } 2\rangle\right)$$

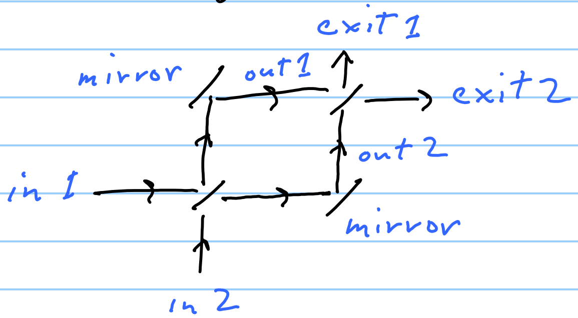
Denoting $a|\text{in } 1\rangle + b|\text{in } 2\rangle$ by column vector $\begin{pmatrix} a \\ b \end{pmatrix} = v_{in}$

and $c|\text{out } 1\rangle + d|\text{out } 2\rangle$ by column vector $\begin{pmatrix} c \\ d \end{pmatrix} = v_{out}$, the out vector $v_{out}$ is related to the in vector $v_{in}$ by

$$V_{out} = \hat{U} V_{in} \quad \text{where} \quad \hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Note that $\hat{U}^\dagger \hat{U} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}$

Equivalently, the rows (or columns) of $\hat{U}$ are mutually orthogonal normalized vectors.




We can build an interferometer by combining two beam splitters and two mirrors as shown, where both beam splitters realize the unitary matrix $\hat{U}$.

Representing $e\,|\text{exit } 1\rangle + f\,|\text{exit } 2\rangle$ by column vector $\begin{pmatrix} e \\ f \end{pmatrix} = V_{exit}$, we have $V_{exit} = \hat{U} V_{out} = \hat{U}\hat{U} V_{in} = \hat{U}^2 V_{in}$

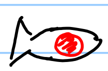
where $\hat{U}^2 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Thus $\hat{U}^2$ maps $|\text{in } 1\rangle \rightarrow |\text{exit } 2\rangle$

$|\text{in } 2\rangle \rightarrow -|\text{exit } 1\rangle$

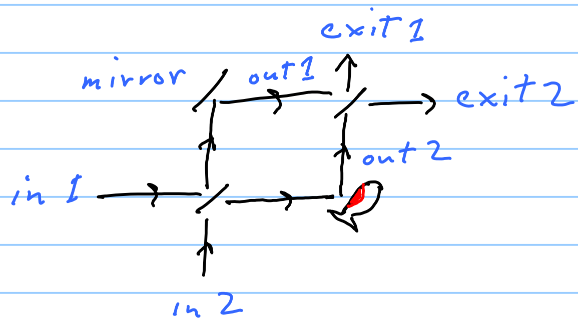
If the input photon enters through port 1, and we place photon detectors at the exit ports, we detect the photon at exit port 1 with probability $P(1) = 0$ and at exit port 2 with probability $P(2) = 1$

Now imagine a bomb with a very sensitive trigger. If a single photon strikes the bomb's target, the photon is absorbed and the bomb explodes!



But the bombs are not perfectly manufactured; some of them are duds. If a bomb is a dud, a photon striking its target is perfectly reflected, and the bomb does not explode.

We want to test the bombs to identify which ones are good and which ones are duds. But how?
We can direct a photon at the target and observe whether the photon bounces back. That's fine if the bomb is a dud, but if the bomb is good the test destroys the bomb. Is there a way to certify that a bomb is good without destroying it?




Well, consider what happens if we replace one of the mirrors in our interferometer with the bomb to be tested.

If the bomb is a dud, it behaves just like the mirror it replaced.

Thus if the input state is $|in\,1\rangle$, then the state at the exit port is $|exit\,2\rangle$.

But if the bomb is good, it realizes a measurement in the basis $\{|out\,1\rangle, |out\,2\rangle\}$. If the photon follows route out 2, the photon is absorbed and the bomb explodes. If the photon follows route out 1, no photon strikes the bomb and it does not explode

If the input state is $|in\,1\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|out\,1\rangle + |out\,2\rangle\right)$, then, the bomb explodes with probability $\frac{1}{2}$. If it does not exploded, the post-measurement state is

$$|out\,1\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|exit\,1\rangle + |exit\,2\rangle\right.$$

Thus for a good bomb, there are 3 possible outcomes:

Bomb explodes, probability $= \frac{1}{2}$,
Photon detected at exit 1, probability $= 1/4$,
Photon detected at exit 2, probability $= 1/4$.

Since when the bomb is a dud the photon is detected at exit 2 with prob $= 1$, if the bomb does not explode and the photon is detected at exit 1 instead, we are certain that the bomb is good, even though no photon ever encountered the bomb!

Of course, with prob $= \frac{1}{2}$ the bomb explodes, so we are sure it is good but we destroyed it by testing it. But with prob $= 1/4$ we manage to certify the bomb is good without exploding it. With prob $= 1/4$ (if the bomb is good) we detect the photon at exit 2. This outcome is inconclusive, and we can repeat the test, trying again to get a conclusive result. What makes the test work is that by blocking the photon passing through route out 2, we can boost the probability of detection at exit 1, from 0 to 1/4.

In the homework, you'll show that there is a more sophisticated version of the test, which boosts the probability of a conclusive outcome to nearly one.

"The Measurement Problem"

Now that we have stated the postulates of quantum mechanics, we note a striking dichotomy: There are two quite different ways in which a quantum state can change:

① Deterministic Unitary evolution.
By solving the Schrödinger equation, we may find the state vector at time $t'$ if we know the state vector at time $t$:

$$|\psi(t')\rangle = U(t',t)\,|\psi(t)\rangle$$

This is reminiscent of classical mechanics, where a unique trajectory passes through each point in phase space. But there is an important difference, too — $|\psi(t)\rangle$ itself is not something we can directly measure; it is not an observable of the theory.

② Probabilistic measurement.

A measurement projects the state vector onto one of a set of mutually orthogonal subspaces. But even if we have a complete description of the state vector right before the measurement, we cannot say with certainty what the measurement outcome will be. Instead,

$$|\psi\rangle \longrightarrow \frac{\hat{E}_n|\psi\rangle}{\|\hat{E}_n|\psi\rangle\|} \quad \text{with probability } P_n = \langle\psi|\hat{E}_n|\psi\rangle,$$

a process called "reduction of the state vector"

But why should there be such a dichotomy? Isn't measurement a physical process like any other and should we therefore be able to describe it with the Schrödinger equation? Is there a way to include both measurement and unitary evolution in a more unified framework?

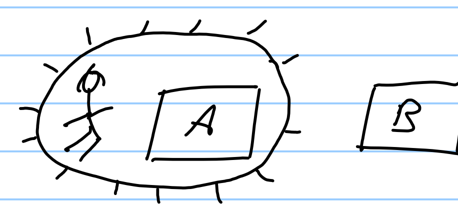
This puzzle is called the "measurement problem" in quantum mechanics. There is not yet a broad consensus in favor of any particular solution to this problem, but we will try to understand the problem more deeply.

## The Density operator

First we will try to understand something else, which may not at first seem to be related to the measurement problem, though it will turn out to be, and in any case it is interesting in its own right.


Our postulates apply to a closed system that does not interact with anything else, like the whole universe. But in practice we usually work with open systems, which do interact with their surroundings. While there may be a state vector that describes the state of a closed system, how should we describe the state of a subsystem, which is a part of this closed system?

Imagine, for example, a system with two parts, labeled A and B, and a physicist (Alice) who is able to observe only part A. What will she see?



We will need one more postulate, telling us how to describe composite systems.

5. <u>Composite systems</u> ("Tensor-product rule")

If system $A$ is described by Hilbert space $\mathcal{H}_A$ and system $B$ by Hilbert space $\mathcal{H}_B$, then the composite system $AB$ is described by the <u>Tensor-product</u> Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

If $\dim \mathcal{H}_A = N_A$ and $\{|e_a\rangle, a = 1, 2, \ldots, N_A\}$ is an ON basis for $\mathcal{H}_A$, and

If $\dim \mathcal{H}_B = N_B$ and $\{|f_i\rangle, i = 1, 2, \ldots, N_B\}$ is an ON basis for $\mathcal{H}_B$,

Then $\dim \mathcal{H}_{AB} = N_A N_B$, and $\left\{|e_a\rangle \otimes |f_i\rangle, \begin{array}{l} a = 1, 2, \ldots, N_A \\ i = 1, 2, \ldots, N_B \end{array}\right\}$ is an ON basis for $\mathcal{H}_{AB}$, where

$$\left(\langle e_a| \otimes \langle f_i|\right)\left(|e_b\rangle \otimes |f_j\rangle\right) = \delta_{ab}\,\delta_{ij}$$

Now, a state vector for $AB$ can be expanded in this tensor product basis:

$$|\psi\rangle = \sum_{a,i} \psi_{ai}\, |e_a\rangle \otimes |f_i\rangle, \text{ where } \langle\psi|\psi\rangle = \sum_{a,i} |\psi_{ai}|^2 = 1.$$

But an observable that Alice can measure acts nontrivially only on part $A$ — it can be expressed as a Tensor product operator $\hat{O} \otimes \hat{I}$,

where $\quad \left(\langle e_a| \otimes \langle f_i|\right) \hat{O} \otimes \hat{I} \left(|e_b\rangle \otimes |f_j\rangle\right) = \langle e_a|\hat{O}|e_b\rangle\, \delta_{ij}$.

Note that we can express

$$|\psi\rangle = \sum_i |\tilde{\psi}_i\rangle \otimes |f_i\rangle \quad \text{where} \quad |\tilde{\psi}_i\rangle = \sum_a \psi_{ai} |e_a\rangle$$

The $\sim$ atop $|\tilde{\psi}_i\rangle$ serves to remind us that $|\tilde{\psi}_i\rangle$ is not a normalized vector. Rather

$$\langle\psi|\psi\rangle = 1 = \sum_i \langle\tilde{\psi}_i|\tilde{\psi}_i\rangle$$

Denoting $\langle\tilde{\psi}_i|\tilde{\psi}_i\rangle = p_i$, we have $\sum_i p_i = 1$

and we may write $|\tilde{\psi}_i\rangle = \sqrt{p_i}\, |\psi_i\rangle$, where $|\psi_i\rangle$ <u>is</u> a normalized vector; however, $|\psi_i\rangle$ and $|\psi_j\rangle$ need not be orthogonal for $i \neq j$.

Now we have, $|\psi\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle \otimes |f_i\rangle$,

and the expectation value of Alice's observable becomes:

$$\langle\psi|\hat{O} \otimes \hat{I}|\psi\rangle = \sum_i p_i \langle\psi_i|\hat{O}|\psi_i\rangle .$$

We could have obtained the same expression for the expectation value in a different situation. Suppose that A were a closed system rather than part of a larger system. And suppose that Alice's state had been chosen by consulting a random number generator, where the state $|\psi_i\rangle$ is prepared with probability $p_i$. Then the expectation value of $\hat{O}$, if we average over <u>both</u> the choice of state <u>and</u> the outcome of Alice's quantum measurement, would also be

$$\sum_i p_i \langle \psi_i | \hat{\mathcal{O}} | \psi_i \rangle$$

exp. of $\mathcal{O}$ if $|\psi_i\rangle$ is chosen

probability $|\psi_i\rangle$ is chosen

These two situations are completely indistinguishable as far as Alice is concerned; there is no measurement she can make that can tell the difference.

In effect, then, when Alice observes part of a larger system, probability enters in two ways. One way is the usual probabilistic nature of quantum measurement; the other is that her state behaves as though it were chosen by sampling from an ensemble of state vectors.

It is convenient to express the expectation value of $\hat{\mathcal{O}} \otimes \hat{I}$ in a different way. Recall that the trace of an operator is defined by

$$\operatorname{tr} \hat{\mathcal{O}} = \sum_a \langle e_a | \hat{\mathcal{O}} | e_a \rangle$$

where $\{|e_a\rangle\}$ is a complete ON basis (and that the trace does not depend on the basis). Thus, we may write (using completeness):

$$\langle \psi | \hat{\mathcal{O}} | \psi \rangle = \sum_a \langle \psi | e_a \rangle \langle e_a | \hat{\mathcal{O}} | \psi \rangle$$

$$= \sum_a \langle e_a | \hat{\mathcal{O}} | \psi \rangle \langle \psi | e_a \rangle = \operatorname{tr}\left[ \hat{\mathcal{O}} \left( |\psi\rangle\langle\psi| \right) \right].$$

And therefore $\langle \hat{\mathcal{O}} \rangle = \sum_i p_i \langle \psi_i | \hat{\mathcal{O}} | \psi_i \rangle = \operatorname{tr}(\hat{\mathcal{O}} \hat{\rho})$,

where $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, which is called the <u>density operator</u> of system A

(Recall that if $|\psi_i\rangle$ is a normalized vector, then $|\psi_i\rangle\langle\psi_i|$ is the projector that projects onto $|\psi_i\rangle$.)

The density operator provides a complete physical description of the state of system $A$, as it encodes the probabilities assigned to all outcomes for any possible measurement we could perform on system $A$.

The density operator has three obvious but important properties:

① It is Hermitian $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \hat{\rho}^\dagger$.

Therefore it has a spectral representation

$$\hat{\rho} = \sum_a |e_a\rangle \lambda_a \langle e_a|$$ where $\{|e_a\rangle\}$ are the eigenvectors, with associated eigenvalues $\{\lambda_a\}$

② Its eigenvalues are nonnegative, because

$$\lambda_a = \langle e_a|\hat{\rho}|e_a\rangle = \sum_i p_i |\langle e_a|\psi_i\rangle|^2 \geq 0$$

③ It has unit trace:

$$\mathrm{tr}\,\hat{\rho} = \sum p_i \,\mathrm{tr}\, |\psi_i\rangle\langle\psi_i| = \sum_i p_i \langle\psi_i|\psi_i\rangle = \sum_i p_i = 1$$

Hence the eigenvalues sum to 1: $\sum_a \lambda_a = 1$.

If the density operator has just one nonzero eigenvalue, it is $\hat{\rho} = |\psi\rangle\langle\psi|$, a projector onto a state vector.

In that case we say the state is <u>pure</u>. Otherwise we say the state is <u>mixed</u>.

Note that ensemble realization of a mixed density is not unique. We can expand the joint state of AB

$$|\psi\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle \otimes |f_i\rangle$$ in terms of a different

basis $\{|g_\mu\rangle\}$ for system B, where

$$|f_i\rangle = \sum_\mu |g_\mu\rangle\langle g_\mu|f_i\rangle = \sum_\mu |g_\mu\rangle V_{\mu i}$$

and $V_{\mu i} = \langle g_\mu|f_i\rangle$ is a unitary matrix

Thus

$$|\psi\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle \otimes \sum_\mu V_{\mu i}\, |g_\mu\rangle = \sum_\mu \sqrt{q_\mu}\, |\varphi_\mu\rangle \otimes |g_\mu\rangle$$

where $$\sqrt{q_\mu}\, |\varphi_\mu\rangle = \sum_i V_{\mu i} \sqrt{p_i}\, |\psi_i\rangle$$

The density operator can be expressed either as

$$\hat{\rho} = \sum_i p_i\, |\psi_i\rangle\langle\psi_i| \quad \text{or} \quad \hat{\rho} = \sum_\mu q_\mu |\varphi_\mu\rangle\langle\varphi_\mu|$$

and can be realized either by selecting $|\psi_i\rangle$ with probability $p_i$ or by selecting $|\varphi_\mu\rangle$ with probability $q_\mu$. Alice can't tell the difference between these two ensembles because both have the same density operator.

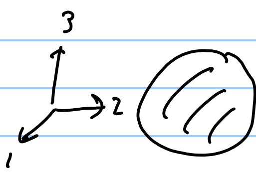
To be concrete, consider the case of a <u>qubit</u>, a two-level system with orthonormal basis $\{|e_0\rangle, |e_1\rangle\}$. What is the general form for the density operator of a qubit? It is a Hermitian $2\times 2$ matrix with trace 1, and can therefore be expressed as

$$\hat{\rho}(\vec{P}) = \frac{1}{2}\begin{pmatrix} 1+P_3 & P_1 - iP_2 \\ P_1 + iP_2 & 1-P_3 \end{pmatrix}$$ where $P_1, P_2, P_3$ are real numbers

But we must also demand that $\hat{\rho}$ has eigenvalues $\geq 0$. Recalling that the determinant of a $2\times 2$ Hermitian matrix is the product of its two eigenvalues, we note that

$$\det\hat{\rho} = \frac{1}{4}\left[(1+P_3)(1-P_3) - (P_1 - iP_2)(P_1 + iP_2)\right]$$
$$= \frac{1}{4}\left(1 - \vec{P}^2\right) \quad \text{where } \vec{P}^2 = P_1^2 + P_2^2 + P_3^2$$

Since $\operatorname{tr}\hat{\rho} = 1$, $\hat{\rho}$ cannot have two negative eigenvalues (which would imply $\operatorname{tr}\hat{\rho} < 0$) so to ensure that the eigenvalues of $\rho$ are nonnegative, it suffices that $\det\hat{\rho} \geq 0$ or $|\vec{P}| \leq 1$.




Therefore, the possible density matrices for a qubit are in 1-1 correspondence with the points of a unit-radius ball in three-dim. space. Pure states (for which $\hat{\rho}$ has eigenvalues 1 and 0, hence $\det\hat{\rho} = 0$) occupy the boundary of the ball, $|\vec{P}| = 1$. We say that the 3-vector $\vec{P}$ is the qubit's "polarization". The ball of possible qubit density operators is called the "Bloch sphere" (even though it is really a ball).

A general pure state for a qubit (up to a physically irrelevant overall phase) can be expressed as

$$|\psi(\theta,\varphi)\rangle = e^{-i\varphi/2}\cos\frac{\theta}{2}\,|e_0\rangle + e^{i\varphi/2}\sin\frac{\theta}{2}\,|e_1\rangle,$$

or as a column vector $|\psi(\theta,\varphi)\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2 \end{pmatrix}$

and the corresponding density operator is

$$\hat{\rho}(\theta,\varphi) = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\theta}{2} \\ e^{i\varphi/2}\sin\theta/2 \end{pmatrix} \left( e^{i\varphi/2}\cos\frac{\theta}{2} \quad e^{-i\varphi/2}\sin\theta/2 \right)$$

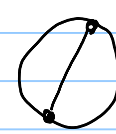
$$= \begin{pmatrix} \cos^2\theta/2 & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\theta/2\cos\theta/2 & \sin^2\theta/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta(\cos\varphi - i\sin\varphi) \\ \sin\theta(\cos\varphi + i\sin\varphi) & 1-\cos\theta \end{pmatrix}.$$

Thus the qubit's polarization is: $P_1 = \sin\theta\cos\varphi$, $P_2 = \sin\theta\sin\varphi$, $P_3 = \cos\theta$

$\theta$ and $\varphi$ are the polar and azimuthal angles of $\vec{P}$, on the surface of the Bloch ball. We can cover the sphere by allowing $\theta$ to vary in the range $\theta\in[0,\pi]$, and $\varphi$ to vary in the range $\varphi\in[0,2\pi]$

(Note that when $\varphi$ increases by $2\pi$, $|\psi\rangle \to -|\psi\rangle$, i.e. $|\psi\rangle$ changes by an overall phase $-1$, which does not change the density operator $\hat{\rho}$.)

Antipodal points on the sphere are mutually orthogonal pure states.



These are the two eigenstates of a certain Hermitian operator

$$\hat{\sigma}(\vec{n}) = \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}$$ where $\vec{n}$ is a unit vector

Notice that $\mathrm{tr}\,\hat{\sigma}(\vec{n}) = 0$ and

$$\hat{\sigma}(\vec{n})^2 = \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}\begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix} = \vec{n}^2\, \hat{I} = \hat{I}$$

so that the eigenvalues of $\sigma(\hat{n})$ are $\pm 1$.

Furthermore,

$$\hat{\sigma}(\vec{n})\,\hat{\rho}(\vec{n}) = \sigma(\vec{n})\,\tfrac{1}{2}\left(\hat{I} + \sigma(\vec{n})\right)$$

$$= \tfrac{1}{2}\left(\hat{\sigma}(\vec{n}) + \hat{I}\right) = \hat{\rho}(\vec{n})$$

and also $\hat{\rho}(\vec{n}) = \hat{\rho}(\vec{n})\,\hat{\sigma}(\vec{n})$.

Thus the pure state $\hat{\rho}(\vec{n}) = |\psi(\hat{n})\rangle\langle\psi(\hat{n})|$ is the eigenstate of $\hat{\sigma}(\vec{n})$ with eigenvalue $+1$.

And $\hat{\sigma}(\vec{n})\,\hat{\sigma}(-\vec{n}) = -\hat{I}$, which implies that

$$\hat{\sigma}(\vec{n})\,\hat{\rho}(-\vec{n}) = \hat{\rho}(-\vec{n})\,\hat{\sigma}(\vec{n}) = -\hat{\rho}(-\hat{n}),$$

i.e. the pure state $\hat{\rho}(-\vec{n})$ is the eigenstate of $\hat{\sigma}(\vec{n})$ with eigenvalue $-1$. We say that $\hat{\rho}(\vec{n})$ and $\hat{\rho}(-\vec{n})$ are the "up" and "down" states along the axis $\vec{n}$. (Sometimes we say "spin-up" and "spin-down" along $\vec{n}$, in honor of an important example of a qubit — the spin of the electron.)

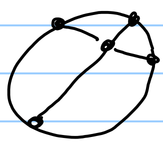
For a general qubit density operator, the expectation value of $\sigma(\vec{n})$ is

$$\langle \sigma(\vec{n}) \rangle_{\vec{P}} = \mathrm{tr}\, \sigma(\vec{n})\, \rho(\vec{P})$$

$$= \mathrm{tr} \begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix} \left[ \frac{1}{2} \hat{I} + \frac{1}{2} \begin{pmatrix} P_3 & P_1 - i P_2 \\ P_1 + i P_2 & -P_3 \end{pmatrix} \right]$$

$$= \frac{1}{2} \left( n_3 P_3 + (n_1 - i n_2)(P_1 + i P_2) + n_3 P_3 + (n_1 + i n_2)(P_1 - i P_2) \right)$$

$$= n_1 P_1 + n_2 P_2 + n_3 P_3 = \vec{n} \cdot \vec{P}.$$



States in the interior of the Bloch sphere can be realized by an ensemble of two pure states.

If $|\psi(\vec{n})\rangle$ with probability $p$

$|\psi(\vec{m})\rangle$ with probability $1-p$,

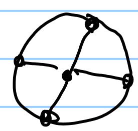
then $\hat{\rho} = p\, |\psi(\vec{n})\rangle\langle\psi(\vec{n})| + (1-p)\, |\psi(\vec{m})\rangle\langle\psi(\vec{m})|$

$$= p\, \hat{\rho}(\hat{n}) + (1-p)\, \hat{\rho}(\vec{m}) = \hat{\rho}\left( p\vec{n} + (1-p)\vec{m} \right),$$

i.e. the polarization is $\vec{P} = p\vec{n} + (1-p)\vec{m}$

This is a point on the straight line connecting the unit vectors $\vec{n}$ and $\vec{m}$. If we choose the line to be the <u>diameter</u> connecting $\vec{n}$ and $-\vec{n}$, then $\hat{\rho}$ is realized by mixing two mutually orthogonal pure states: $\hat{\rho}(\vec{n})$ and $\hat{\rho}(-\vec{n})$. But there are also many ways to realize $\hat{\rho}$ as a mixture of nonorthogonal states - we can mix the states at the endpoints of any chord of the Bloch sphere that contains the point $\vec{P}$.

The state with polarization vector $\vec{P}=0$, $\hat{\rho}=\frac{1}{2}\hat{I}$, is said to be "maximally mixed."



It can be realized by an ensemble of equiprobable mutually orthogonal pure states:

$$\hat{\rho}(\vec{0})=\frac{1}{2}\hat{\rho}(\vec{n})+\frac{1}{2}\hat{\rho}(-\vec{n})$$

In this case, all chords through $\vec{P}=\vec{0}$ are diameters, and we can realize the density operator as a equal mixture of mutually orthogonal states in many different ways. All of these realizations are physically equivalent, in that they yield the same expectation values for all possible measured observables.