1. **A barrier in a well** (10 points).

A free quantum-mechanical particle with mass $m$ moves inside a one-dimensional box with impenetrable walls located at $x = \pm a$. Furthermore, a repulsive $\delta$-function barrier sits at the center of the well, so the potential energy function $V(x)$ in between the two impenetrable walls is given by

$$V(x) = \frac{\hbar^2}{m} \Delta \delta(x),$$

where $\delta(x)$ denotes the Dirac $\delta$-function and $\Delta > 0$. As explained in Problem 2 last week, this $\delta$-function potential causes the logarithmic derivative of the wave function $\varphi(x)$ to jump discontinuously at the origin:

$$\varphi'(0^+) - \varphi'(0^-) = 2\Delta \varphi(0).$$

Here $\varphi'(x)$ denotes the first derivative of $\varphi(x)$, and $\varphi'(0^+)$ (respectively $\varphi'(0^-)$) denotes the limit of $\varphi'(x)$ as $x$ approaches zero from positive (negative) values. The sign convention used here for $\Delta$ is the opposite of that used last week; now $\Delta > 0$ is the case of a repulsive barrier.

*a) For the even energy eigenstates, what is the value of the logarithmic derivative $\varphi'(x)/\varphi(x)$ at $x = 0^+$ and $x = 0^-$?*

*b) For the even energy eigenstates, derive an equation that determines the wavenumber $k$ implicitly, where $E = \hbar^2 k^2/2m$. Express your answer in the form

$$\Delta a = f(ka),$$

where $f$ is a suitable function.*

*c) Consider the limiting case of an infinitely strong repulsive barrier: $\Delta a \to \infty$. What are values of the energy eigenvalues in this limit, for both even and odd $n$?*

*d) Draw rough sketches of the wave functions for the ground state and the first excited state in the limit $\Delta a \to \infty$.*
2. Reflectionless potential (15 points).

Consider a particle with mass $m$ moving in the attractive potential

$$V(x) = -\frac{\hbar^2 k_0^2}{m} \operatorname{sech}^2(k_0 x),$$

where $\operatorname{sech}(z) = 2 (e^z + e^{-z})^{-1}$ denotes the hyperbolic secant function.

a) Show that the time-independent Schrödinger equation for this potential can be expressed as

$$\left(-\frac{d^2}{dz^2} - 2 \operatorname{sech}^2(z)\right) \varphi(z) = \tilde{k}^2 \varphi(z), \quad (1)$$

where $z = k_0 x$ is a dimensionless position variable, and $\tilde{k}^2 k^2 / k_0^2 = 2mE/\hbar^2 k_0^2$ is a dimensionless wavenumber.

b) Show that

$$(i\tilde{k} - \tanh(z)) e^{ikz} \quad (2)$$

solves eq.(1).

c) Show that eq.(2) approaches $A e^{ikz}$ as $z \to -\infty$ and approaches $C e^{ikz}$ as $z \to +\infty$, where $A$ and $C$ are constants. What are the values of these constants.

d) What is the transmission amplitude $C/A$? Show that the transmission probability $T = |C/A|^2$ is one, and that the reflection probability $R = 1 - T$ is zero. Hence, if a wavepacket is incident on this potential from the far left, there is no reflected wave packet at all.

e) Find an imaginary value $\tilde{k} = i\kappa$ such that $A/C = 0$ and the transmission amplitude thus diverges. For this value of $\kappa$, there is a normalizable bound state solution, which decays exponential for both $z \to +\infty$ and $z \to -\infty$.

f) Check that

$$\varphi(z) = \operatorname{sech}(z)$$

solves eq.(1), where $\tilde{k}^2 = -\kappa^2$ and $\kappa$ is the value found in (e). This is the bound state solution. What is the corresponding bound state energy?
3. **Bound states in a linear potential** (15 points).

Consider a particle with mass \( m \) moving in the potential

\[
V(x) = F|x|
\]

where \(|x|\) denotes the absolute value function. Thus there is a constant force \( F \) directed toward the origin.

**a)** Show that the time-independent Schrödinger equation for this potential can be expressed in the form (for \( x \geq 0 \))

\[
\left(-\frac{d^2}{dy^2} + y\right) \varphi(y) = \tilde{E} \varphi(y),
\]

where

\[
y = \left(\frac{\hbar^2}{2mF}\right)^{-1/3} x, \quad \tilde{E} = \left(\frac{\hbar^2 F^2}{2m}\right)^{-1/3} E.
\]

Equivalently, we may write

\[
\frac{d^2}{dz^2} \varphi(z) = z \varphi(z)
\]

where \( z = y - \tilde{E} \). The solution to this equation that decays as \( z \to +\infty \) is the Airy function \( \text{Ai}(z) \).

All real zeros of \( \text{Ai}(z) \) and of its first derivative \( \text{Ai}'(z) \) occur for \( z < 0 \). We denote the zeros of \( \text{Ai}'(z) \), in order of increasing absolute value, by \( a_0, a_2, a_4, \ldots \), and we denote the zeros of \( \text{Ai}(z) \) in order of increasing absolute value by \( a_1, a_3, a_5, \ldots \). These constants have the numerical values:

\[
-a_0 = 1.0188 \ldots \\
-a_1 = 2.3381 \ldots \\
-a_2 = 3.2482 \ldots \\
-a_3 = 4.0879 \ldots \\
-a_4 = 4.8201 \ldots \\
-a_5 = 5.5206 \ldots 
\]
b) Show that for \( n = 0, 1, 2, \ldots \) there is a bound state solution to the Schrödinger equation with \( n \) nodes and dimensionless “energy” \( \bar{E} = E_n = -a_n \).

Using the WKB approximation and the connection formulas, we can derive the Bohr-Sommerfeld criterion:

\[
\int_{x_1}^{x_2} dx \ k(x) = \pi \left( n + \frac{1}{2} \right),
\]

where \( n \) is the number of nodes in the bound state wavefunction, \( E_n \) is the corresponding energy, \( x_1 \) and \( x_2 \) are the classical turning points for \( E = E_n \), and \( k(x)^2 = 2m (E_n - V(x)) / \hbar^2 \). For the harmonic potential, this WKB estimate actually agrees with the exact value of \( E_n \), but in general there are corrections higher order in \( 1/n \).

c) Apply the WKB criterion to the linear potential, deriving a formula for \( \bar{E}_n \). For \( n = 0, 1, 2, 3, 4, 5 \), compare to the exact result from (b). You should find pretty good agreement for all \( n \geq 1 \). Furthermore, you should find (considering the odd and even values of \( n \) separately), that the agreement gets systematically better as \( n \) increases.