1. Weaker decoherence. In class we discussed the phase damping of a qubit that results when the qubit scatters a photon with probability $p$. The scattered photon is knocked into one of two mutually orthogonal states $\{ |0\rangle_E, |1\rangle_E \}$, correlated with the qubit’s state, both of which are orthogonal to the state $|\text{un}\rangle_E$ of the unscattered photon. If the initial state of the qubit is $|\psi\rangle_S = a|0\rangle_S + b|1\rangle_S$, then the joint state of the qubit and photon evolves as

\[
|\psi\rangle_S \otimes |\text{un}\rangle_E \rightarrow \sqrt{1-p} |\psi\rangle_S \otimes |\text{un}\rangle_E + \sqrt{p} (a|0\rangle_S \otimes |0\rangle_E + b|1\rangle_S \otimes |1\rangle_E).
\]

Thus the qubit density operator $\hat{\rho}$ evolves as

\[
\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \rightarrow \hat{\rho}' = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}.
\]

Now consider a different model of decoherence, in which photon scattering does not perfectly resolve the state of the qubit. The scattered photon is knocked to the normalized state $|\gamma\rangle_E$ if the qubit’s state is $|0\rangle_S$ and it is knocked to the normalized state $|\eta\rangle_E$ if the photon’s state is $|1\rangle_S$; thus eq.(1) is replaced by

\[
|\psi\rangle_S \otimes |\text{un}\rangle_E \rightarrow \sqrt{1-p} |\psi\rangle_S \otimes |\text{un}\rangle_E + \sqrt{p} (a|0\rangle_S \otimes |\gamma\rangle_E + b|1\rangle_S \otimes |\eta\rangle_E).
\]

Both $|\gamma\rangle_E$ and $|\eta\rangle_E$ are orthogonal to the state $|\text{un}\rangle_E$ of the unscattered photon, but they are not necessarily mutually orthogonal; rather

\[
E\langle \eta|\gamma\rangle_E = 1 - \epsilon,
\]

where $\epsilon$ is a real number. Thus for $\epsilon = 1$, the states $|\gamma\rangle_E$ and $|\eta\rangle_E$ are orthogonal, and we recover the model considered previously, while for $\epsilon = 0$, the scattered photon remains uncorrelated with the qubit, and there is no decoherence at all.

Show that eq.(2) implies that the density operator evolves according to

\[
\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \rightarrow \hat{\rho}' = \begin{pmatrix} \rho_{00} & \lambda \rho_{01} \\ \lambda \rho_{10} & \rho_{11} \end{pmatrix},
\]
and express $\lambda$ in terms of $p$ and $\epsilon$. You may find it convenient to expand $|\gamma\rangle_E$ and $|\eta\rangle_E$ in terms of two orthonormal vectors $|e_0\rangle_E$ and $|e_1\rangle_E$ (which are both orthogonal to $|un\rangle_E$), so that

$$|\gamma\rangle_E = \gamma_0|e_0\rangle_E + \gamma_1|e_1\rangle_E, \quad |\eta\rangle_E = \eta_0|e_0\rangle_E + \eta_1|e_1\rangle_E.$$ 

2. Master equation for spontaneous decay. A two-level atom has a ground state $|g\rangle$ with energy $E_g$ and an excited state $|e\rangle$ with energy $E_e = E_g + \hbar\omega$. We may adjust the definition of energy by an additive constant, so that the ground state has zero energy, and the excited state’s energy is $\hbar\omega$. Thus the Hamiltonian of the atom is

$$\hat{H} = \hbar\omega|e\rangle\langle e|,$$

and the state vector $|\psi(t)\rangle$ of the atom evolves according to

$$|\psi(t+dt)\rangle = |\psi(t)\rangle - i\omega dt|e\rangle\langle e|\psi\rangle,$$

where $dt$ is an infinitesimal time increment.

a) Recall that a general density operator $\hat{\rho}$ for the atom can be represented as $\hat{\rho} = \sum_a p_a |\psi_a\rangle\langle \psi_a|$, where each $|\psi_a\rangle$ is a normalized state vector and the $p_a$’s are positive real numbers satisfying $\sum p_a = 1$. Show that the time-evolving density operator $\hat{\rho}(t)$ obeys the differential equation

$$\frac{d}{dt}\hat{\rho} = -i\omega|e\rangle\langle e|\hat{\rho} + i\omega\hat{\rho}|e\rangle\langle e|.$$ 

(3)

b) Writing the density operator as the matrix

$$\hat{\rho} = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix},$$

in the basis $\{|g\rangle, |e\rangle\}$, express eq.(3) as four separate differential equations for $\rho_{gg}(t) = \langle g|\hat{\rho}|g\rangle$, $\rho_{ge}(t) = \langle g|\hat{\rho}|e\rangle$, $\rho_{eg}(t) = \langle e|\hat{\rho}|g\rangle$, $\rho_{ee}(t) = \langle e|\hat{\rho}|e\rangle$. Solve these equations, finding $\hat{\rho}(t)$ in terms of $\hat{\rho}(0)$.

Now suppose that the excited state of the atom can decay to the ground state by emitting a photon. The rate for the decay process is
Thus in the time interval \((t, t + dt)\) the joint state of the atom and its environment evolves according to
\[
|\psi(t)\rangle \otimes |0\rangle \rightarrow \left( |g\rangle \langle g| |\psi(t)\rangle + \sqrt{1 - \Gamma dt} |e\rangle \langle e| |\psi(t)\rangle \right) \otimes |0\rangle + \sqrt{\Gamma dt} |g\rangle \langle e| |\psi(t)\rangle \otimes |1\rangle.
\]
Here \(|0\rangle\) denotes the state of the environment containing no photon, and \(|1\rangle\) denotes the state of the environment containing one photon.

(F or now we are considering only the evolution due to spontaneous decay, we are ignoring the evolution arising from the Hamiltonian \(\hat{H}.\)

\(c)\) Show that eq.(4) implies a differential equation satisfied by the density operator
\[
\frac{d}{dt}\hat{\rho} = \Gamma|g\rangle \langle e| \hat{\rho} |e\rangle \langle g| - \frac{1}{2} \Gamma|e\rangle \langle e| \hat{\rho} - \frac{1}{2} \Gamma \hat{\rho} |e\rangle \langle e|.
\]
Eq.(5) is called the atom’s master equation.

\(d)\) Extract from eq.(5) differential equations satisfied by \(\rho_{ee}, \rho_{eg},\) and \(\rho_{ge}.\) Solve these equations, finding \(\hat{\rho}(t)\) in terms of \(\hat{\rho}(0).\)

\(e)\) When we combine eq.(5) describing spontaneous decay with eq.(3) describing the atom’s evolution governed by the Schrödinger equation, we obtain a new master equation
\[
\frac{d}{dt}\hat{\rho} = \left( -i\omega - \frac{1}{2} \Gamma \right) |e\rangle \langle e| \hat{\rho} + \left( i\omega - \frac{1}{2} \Gamma \right) \hat{\rho} |e\rangle \langle e| + \Gamma |g\rangle \langle e| \hat{\rho} |e\rangle \langle g|.
\]
Again, find differential equations satisfied by \(\rho_{ee}, \rho_{eg},\) and \(\rho_{ge}\) and solve them, determining \(\hat{\rho}(t)\) in terms of \(\hat{\rho}(0).\)

3. Diagonalizing the density operator. Suppose that the state of a qubit is prepared by flipping a fair coin, and then preparing the state vector \(|\psi_H\rangle\) if the outcome of the coin flip is heads, and preparing the state vector \(|\psi_T\rangle\) if the outcome of the coin flip is tails, where
\[
|\psi_H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_T\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

\(a)\) Find the density operator \(\hat{\rho}\) of the qubit.

\(b)\) What are the eigenvalues of \(\hat{\rho}\)?
c) Find the eigenvectors of \( \hat{\rho} \). It is convenient to express your answer in terms of \( \cos(\pi/8) \) and \( \sin(\pi/8) \).

4. **Schmidt decomposition.** Consider a composite quantum system \( AB \) with Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \), where

\[
\text{dimension } (\mathcal{H}_A) = \text{dimension } (\mathcal{H}_B) = N.
\]

If \( \{|e_a\}, a = 1, 2, 3, \ldots N \rangle \) is an orthonormal basis for \( \mathcal{H}_A \) and and \( \{|f_b\}, b = 1, 2, 3, \ldots N \rangle \) is an orthonormal basis for \( \mathcal{H}_B \), then a normalized state vector \( \psi \rangle \in \mathcal{H}_{AB} \) can be expanded as

\[
\psi = \sum_{a,b=1}^{N} \psi_{ab} |e_a \rangle \otimes |f_b \rangle,
\]

where

\[
\sum_{a,b=1}^{N} |\psi_{ab}|^2 = 1.
\]

a) Any \( N \times N \) matrix \( M \) has a singular value decomposition \( M = UDV \), where \( U \) and \( V \) are \( N \times N \) unitary matrices, and \( D \) is a diagonal real \( N \times N \) matrix with nonnegative eigenvalues. Use the singular value decomposition to show that for any given state vector \( \psi \rangle \in \mathcal{H}_{AB} \), one can choose an orthonormal basis \( \{|e'_a\}, a = 1, 2, 3, \ldots N \rangle \) for \( \mathcal{H}_A \) and an orthonormal basis \( \{|f'_b\}, b = 1, 2, 3, \ldots N \rangle \) for \( \mathcal{H}_B \) such that \( \psi \rangle \) can be expressed as

\[
\psi = \sum_{a=1}^{N} \sqrt{p_a} |e'_a \rangle \otimes |f'_a \rangle,
\]

where the \( \{p_a\} \) are nonnegative real numbers such that

\[
\sum_{a=1}^{N} p_a = 1.
\]

This expression is called the Schmidt decomposition of the state vector \( \psi \rangle \).

b) Using the expression for \( \psi \rangle \) in eq.(7), express the density operator \( \hat{\rho}_A \) for system \( A \) in the basis \( \{|e'_a\} \) and express the density operator \( \hat{\rho}_B \) for system \( B \) in the basis \( \{|f'_a\} \).

c) What are the eigenvalues of \( \hat{\rho}_A \) and of \( \hat{\rho}_B \)?