1. **Quantized rotor.** A wheel spinning in a plane can be described as a Hamiltonian dynamical system with one degree of freedom: the coordinate is the angular orientation $\theta$ taking values in the interval $[0, 2\pi)$, and the conjugate momentum is the angular momentum $L$. The Hamiltonian $H$ is
\[ H = \frac{L^2}{2I}, \]
where $I$ is the moment of inertia.

- **a)** What are the Hamilton equations of motion for this system? Is there a conserved constant of the motion? What is the associated symmetry?

In quantum mechanics, the Hilbert space for this system is the space of square-integrable periodic functions of $\theta$, i.e. functions with the properties
\[ \psi(\theta + 2\pi) = \psi(\theta), \quad \int_0^{2\pi} d\theta |\psi(\theta)|^2 < \infty. \]

The angular momentum operator becomes
\[ \hat{L} = -i\hbar \frac{d}{d\theta}. \]

- **b)** Find the eigenvalues and normalized eigenfunctions of the operator $\hat{L}$. That is, find all values of $\lambda$ and functions $\psi_{\lambda}(\theta)$ such that
\[ \hat{L}\psi_{\lambda}(\theta) = \lambda\psi_{\lambda}(\theta), \quad \psi_{\lambda}(\theta + 2\pi) = \psi_{\lambda}(\theta), \quad \int_0^{2\pi} d\theta |\psi_{\lambda}(\theta)|^2 = 1. \]

- **c)** Verify that the eigenfunctions with distinct eigenvalues are mutually orthogonal:
\[ \int_0^{2\pi} d\theta \psi_{\lambda}(\theta)^* \psi_{\lambda'}(\theta) = 0 \quad \text{for} \quad \lambda \neq \lambda'. \]

- **d)** What are the eigenvalues and eigenfunctions of the Hamiltonian $\hat{H} = \hat{L}^2/2I$?
e) The expectation value of the angular momentum is

\[ \langle \hat{L} \rangle = \int_{0}^{2\pi} d\theta \, \psi(\theta)^* \hat{L} \psi(\theta). \]

Show that if the wavefunction \( \psi(\theta) \) is real (i.e. \( \psi(\theta) = \psi(\theta)^* \)), then \( \langle \hat{L} \rangle = 0 \).

2. **Twisted rotor.** Now consider a nonstandard way to quantize the spinning wheel — the wavefunction \( \psi(\theta) \) is not periodic, but instead “periodic up to a phase”:

\[ \psi(\theta + 2\pi) = e^{i\alpha} \psi(\theta), \]

where \( e^{i\alpha} \) is a fixed complex number with modulus one. For this “twisted rotor,” repeat parts (b)–(d) of Problem 1.

3. **More eigenfunctions.** For square-integrable functions on the real line, consider the Hermitian operator

\[ \hat{H} = -\frac{d^2}{dx^2} + x^2. \]

a) Show that the functions

\[ \psi_0(x) = e^{-x^2/2}, \quad \psi_1(x) = xe^{-x^2/2} \]

are eigenfunctions of \( \hat{H} \), and find their eigenvalues. Check that \( \psi_0(x) \) and \( \psi_1(x) \) are orthogonal functions.

b) Find a real value of \( C \) such that

\[ \psi_2(x) = (x^2 + C)e^{-x^2/2} \]

is an eigenfunction of \( \hat{H} \), and find its eigenvalue.

c) Check that \( \psi_2 \) is orthogonal to \( \psi_0 \) and \( \psi_1 \). It’s useful to recall that

\[ \int_{-\infty}^{\infty} dx \, e^{-x^2} = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} dx \, x^2 e^{-x^2} = \sqrt{\pi}/2. \]

4. **The qubit.** A qubit is a quantum system whose Hilbert space is two dimensional; linear operators acting on a qubit are \( 2 \times 2 \) matrices.
a) Show that the most general Hermitian operator acting on a qubit can be expressed as

\[ a\hat{I} + b\hat{\sigma}(\theta, \phi) \]

where

\[ \hat{\sigma}(\theta, \phi) = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}. \]

Here \( \hat{I} \) is the 2 \( \times \) 2 identity matrix, \( a \) is an arbitrary real number, \( b \) is a nonnegative real number, \( \theta \) is a real number in the interval \([0, \pi]\), and \( \phi \) is a real number in the interval \([0, 2\pi]\).

b) Find the eigenvalues and eigenvectors of the operator \( \hat{\sigma}(\theta, \phi) \). It is convenient to express the eigenvectors in terms of \( \cos(\theta/2) \), \( \sin(\theta/2) \), \( e^{i\phi/2} \) and \( e^{-i\phi/2} \).

The 2 \( \times \) 2 Pauli spin matrices are the Hermitian operators

\[ \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

c) For both of the eigenvectors found in (b), evaluate the expectation values

\[ \langle \hat{\sigma}_1 \rangle, \quad \langle \hat{\sigma}_2 \rangle, \quad \langle \hat{\sigma}_3 \rangle. \]

It is convenient to express the expectation values in terms of \( \cos \theta \), \( \sin \theta \), \( \cos \phi \), \( \sin \phi \).

d) If the observable \( \hat{\sigma}_3 \) is measured, the outcome can be either one of its eigenvalues, +1 or −1. For both of the eigenvectors found in part (b), find the probability \( P(+) \) for the +1 outcome of a \( \hat{\sigma}_3 \) measurement and the probability \( P(−) \) for the −1 outcome of a \( \hat{\sigma}_3 \) measurement.