Ph 12b Midterm Exam
Due: Wednesday, 10 February 2010, 5pm

• This exam is to be taken in one continuous time interval not to exceed 3 hours, beginning when you first open the exam.

• You may consult the textbook *Introductory Quantum Mechanics* by Liboff, the textbook *Introduction to Quantum Mechanics* by Griffiths, your lecture notes, the online lecture notes, and the problem sets and solutions. If you wish, you may use a calculator, computer, or integral table for doing calculations. However, this probably won’t be necessary. **No other materials or persons are to be consulted.**

• There are three problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.

• The completed exam is to be handed in at the Ph 12 in-box outside 264 Lauritsen. All exams are due at 5pm on Wednesday, February 10. **No late exams will be accepted.**

• Good luck!
1. Two-state quantum dynamics — 35 total points

Let \( |e_1\rangle \) and \( |e_2\rangle \) denote two normalized and mutually orthogonal states in a Hilbert space \( \mathcal{H} \): 
\[ \langle e_1 | e_1 \rangle = \langle e_2 | e_2 \rangle = 1, \quad \langle e_1 | e_2 \rangle = 0. \]
A certain quantum system has Hamiltonian \( \hat{H} \), and the two normalized states 
\[ |\omega_1\rangle = \frac{1}{2} |e_1\rangle + \frac{\sqrt{3}}{2} |e_2\rangle, \]
\[ |\omega_2\rangle = -\frac{\sqrt{3}}{2} |e_1\rangle + \frac{1}{2} |e_2\rangle, \]
are eigenstates of \( \hat{H} \) with eigenvalues \( \hbar \omega_1, \hbar \omega_2 \) respectively:
\[ \hat{H} |\omega_1\rangle = \hbar \omega_1 |\omega_1\rangle, \]
\[ \hat{H} |\omega_2\rangle = \hbar \omega_2 |\omega_2\rangle. \]

At time \( t = 0 \), the system is prepared in the state \( |\psi(0)\rangle = |e_1\rangle \).

(a) (10 points) Express \( |\psi(0)\rangle \) as a linear combination of energy eigenstates.

(b) (5 points) Solve the time-dependent Schrödinger equation
\[ i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \]
to find the state \( |\psi(t)\rangle \) at time \( t \). Express your answer in the form 
\[ |\psi(t)\rangle = f_1(t) |\omega_1\rangle + f_2(t) |\omega_2\rangle. \]

(c) (10 points) Now re-express \( |\psi(t)\rangle \) in the form 
\[ |\psi(t)\rangle = g_1(t) |e_1\rangle + g_2(t) |e_2\rangle. \]

(d) (10 points) The states \( |e_1\rangle \) and \( |e_2\rangle \) are eigenstates of an observable \( \hat{A} \):
\[ \hat{A} |e_1\rangle = a_1 |e_1\rangle, \]
\[ \hat{A} |e_2\rangle = a_2 |e_2\rangle, \]
where \( a_1 \neq a_2 \), that is measured at time \( t \). Find the probability \( P(a_1) \) that the outcome of the measurement is \( a_1 \) and the probability \( P(a_2) \) that the outcome is \( a_2 \).

2. Particle in a box — 35 total points

A free quantum-mechanical particle with mass \( m \) moves inside a one-dimensional box with impenetrable walls located at \( x = \pm a/2 \).

(a) (10 points) Find the normalized wave function \( \psi_0(x) \) of the energy eigenstate of lowest energy, and the normalized wave function \( \psi_1(x) \) of the energy eigenstate of next-to-lowest energy. Find also the corresponding energy eigenvalues \( E_0 \) and \( E_1 \).
(b) (5 points) Suppose that at time $t = 0$ the particle is in the state with wave function $\psi(x, 0) = \sqrt{\frac{1}{3}} \psi_0(x) + \sqrt{\frac{2}{3}} \psi_1(x)$. What is the wave function $\psi(x, t)$ at the subsequent time $t$?

(c) (20 points) Find the expectation value $\langle \psi(t) | \hat{x} | \psi(t) \rangle$ of the position operator $\hat{x}$ at time $t$. (Hint: An integral of the form $\int dx \ x \sin(Ax)$ can be done using integration by parts.)

3. Two qubits — 30 total points

A qubit is a quantum system whose Hilbert space is two dimensional. Consider two qubits labeled $A$ and $B$, where $\{|e_0\rangle, |e_1\rangle\}$ is a basis for qubit $A$, and $\{|f_0\rangle, |f_1\rangle\}$ is a basis for qubit $B$. Suppose the state vector for the composite system $AB$ is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( \cos \theta |e_0\rangle \otimes |f_0\rangle + \sin \theta |e_0\rangle \otimes |f_1\rangle + \sin \theta |e_1\rangle \otimes |f_0\rangle + \cos \theta |e_1\rangle \otimes |f_1\rangle \right).$$

(a) (15 points) Alice has no access to qubit $B$; she can perform measurements only on qubit $A$. For any measurement that Alice might perform on qubit $A$, the probability distribution for the measurement outcomes is determined by the density operator $\hat{\rho}$ for qubit $A$. Express this density operator as a $2 \times 2$ matrix in the basis $\{|e_0\rangle, |e_1\rangle\}$.

(b) (5 points) Suppose Alice performs an orthogonal measurement in the basis $\{|e_0\rangle, |e_1\rangle\}$. What is the probability that Alice’s outcome is $|e_0\rangle$?

(c) (10 points) Now suppose that Bob, who does have access to qubit $B$, measures his qubit in the basis $\{|f_0\rangle, |f_1\rangle\}$, and reports to Alice that he obtained the outcome $|f_0\rangle$. After learning of Bob’s outcome, Alice performs her measurement in the basis $\{|e_0\rangle, |e_1\rangle\}$. In this case, what is the probability that Alice’s outcome is $|e_0\rangle$?