(3)

$$M_1(x,y) = x + y$$
$$M_2(x,y) = y + f(x+y)$$

a)
$$DM = \begin{pmatrix} \partial M_1/\partial x & \partial M_1/\partial y \\ \partial M_2/\partial x & \partial M_2/\partial y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ f'(x+y) & 1+f'(x+y) \end{pmatrix}$$

$$\det(DM) = 1 + f'(x+y) - f'(x+y) = 1$$

$\Longrightarrow$ The map is area preserving.

b) Fixed point:

$x + y = x \Longrightarrow y = 0$

$y + f(x+y) = y \Longrightarrow f(x+y) = 0 \Rightarrow x + y = 0 \Rightarrow x = 0$

Fixed point is at $x = y = 0$.

At the fixed point: $DM = \begin{pmatrix} 1 & 1 \\ K & 1+K \end{pmatrix}$ where $K = f'(0)$

If λ is an eigenvalue of DM, then

$$(\lambda - 1)(\lambda - 1 - K) - K = 0 \Rightarrow \lambda^2 - (K+2)\lambda + 1 = 0$$

Solve: $\lambda = \frac{1}{2}\left[K + 2 \pm \sqrt{(K+2)^2 - 4}\right]$

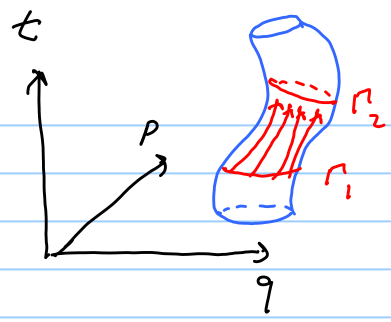
$$\Longrightarrow \lambda = 1 + \frac{K}{2} \pm \frac{1}{2}\sqrt{K(K+4)}$$

The fixed point is elliptic if eigenvalues are complex and hyperbolic if eigenvalues are real.

Elliptic: $-4 < K < 0$

Hyperbolic: $K > 0$ or $K < -4$

(4)



$$dp_i \wedge dq_i = \left(\frac{\partial p_i}{\partial s} \frac{\partial q_i}{\partial t} - \frac{\partial q_i}{\partial s} \frac{\partial p_i}{\partial t} \right) ds \wedge dt$$

using antisymmetry of wedge product

$$= \left(\frac{\partial p_i}{\partial s} \frac{\partial H}{\partial p_i} + \frac{\partial q_i}{\partial s} \frac{\partial H}{\partial q_i} \right) ds \wedge dt$$

using Hamilton's equations

$$dH \wedge dt = \left(\frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial s} ds + \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial s} ds \right) \wedge dt$$

$$= dp_i \wedge dq_i$$ — as was to be shown.

Solutions to problems 1 and 2

*** Use Energy = 1 / 12 to find the initial value x '[0] : ***

```
N[(1 / 6 - (.05) ^2 - (.1) ^2 - (.2) ^2 - 2 * (.1) ^2 * (-0.2) + 2 * (-0.2) ^3 / 3) ^.5, 10]
```

```
0.335907
```

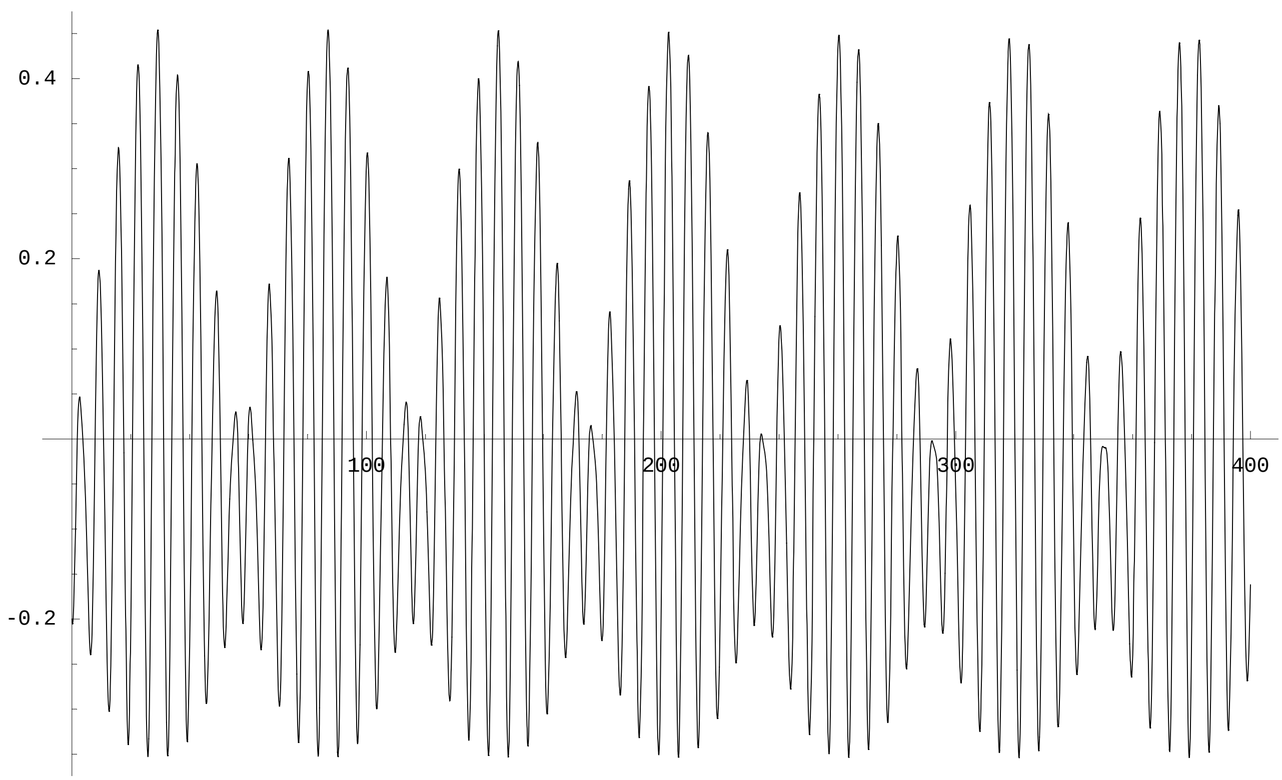
*** For specified initial data,**
solve coupled second - order differential equations for x[t] and y[t] : *

```
NDSolve[{x''[t] == -x[t] - 2 x[t] * y[t], y''[t] == -y[t] - x[t]^2 + y[t]^2, x[0] == -0.1,
  y[0] == -0.2, x'[0] == 0.3359067330872266`, y'[0] == -0.05}, {x, y}, {t, 0, 1000}]
```

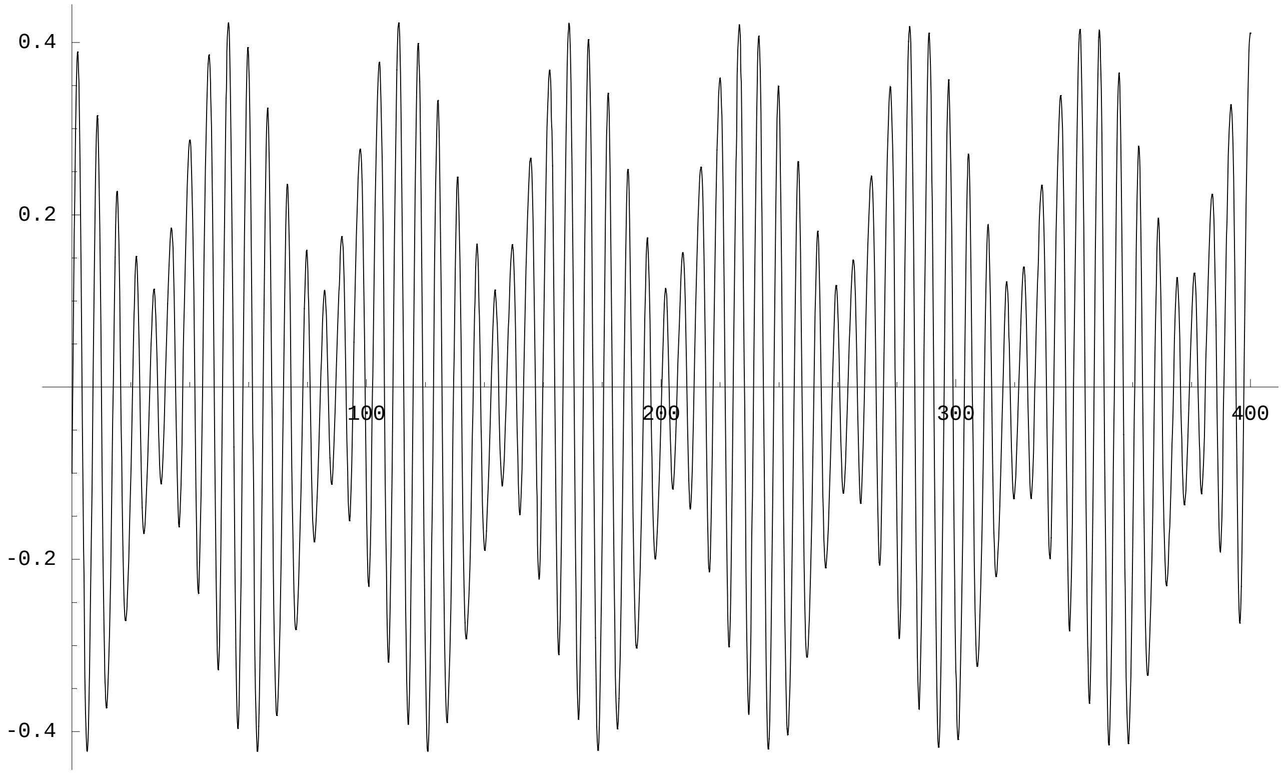
```
{{x → InterpolatingFunction[{{0., 1000.}}, <>],
  y → InterpolatingFunction[{{0., 1000.}}, <>]}}
```

*** Plot for t between 0 and 400. Note that motion is regular : ***

```
Plot[Evaluate[y[t] /. %3], {t, 0, 400}, PlotPoints → 1000, ImageSize → {600, 400}];
```

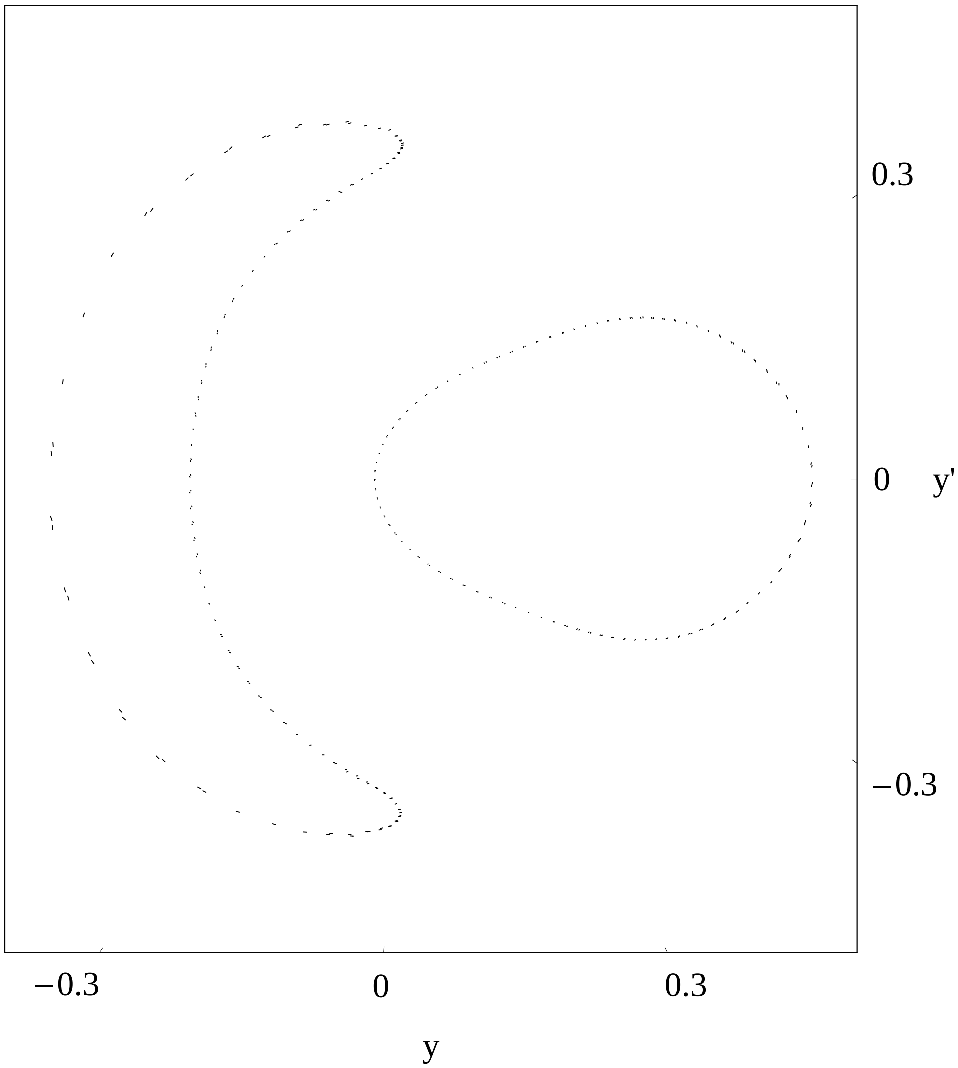


```
Plot[Evaluate[x[t] /. %3], {t, 0, 400}, PlotPoints → 1000, ImageSize → {600, 400}];
```




```
* Plot the Poincare section. Points lie on invariant curves : *
```

```
ParametricPlot3D[Evaluate[{x[t], y[t], y'[t]} /. %3], {t, 0, 1000}, PlotPoints → 4000,
  PlotRange → {{0, 0.001}, {-.4, .5}, {-.5, .5}}, ViewPoint -> {1, 0, 0},
  AxesLabel → {" ", "y", "y'"}, Ticks → {{ }, {-.3, 0, .3}, {-.3, 0, .3}},
  TextStyle → {FontFamily -> "Times", FontSize → 16}, ImageSize → {500, 500}];
```



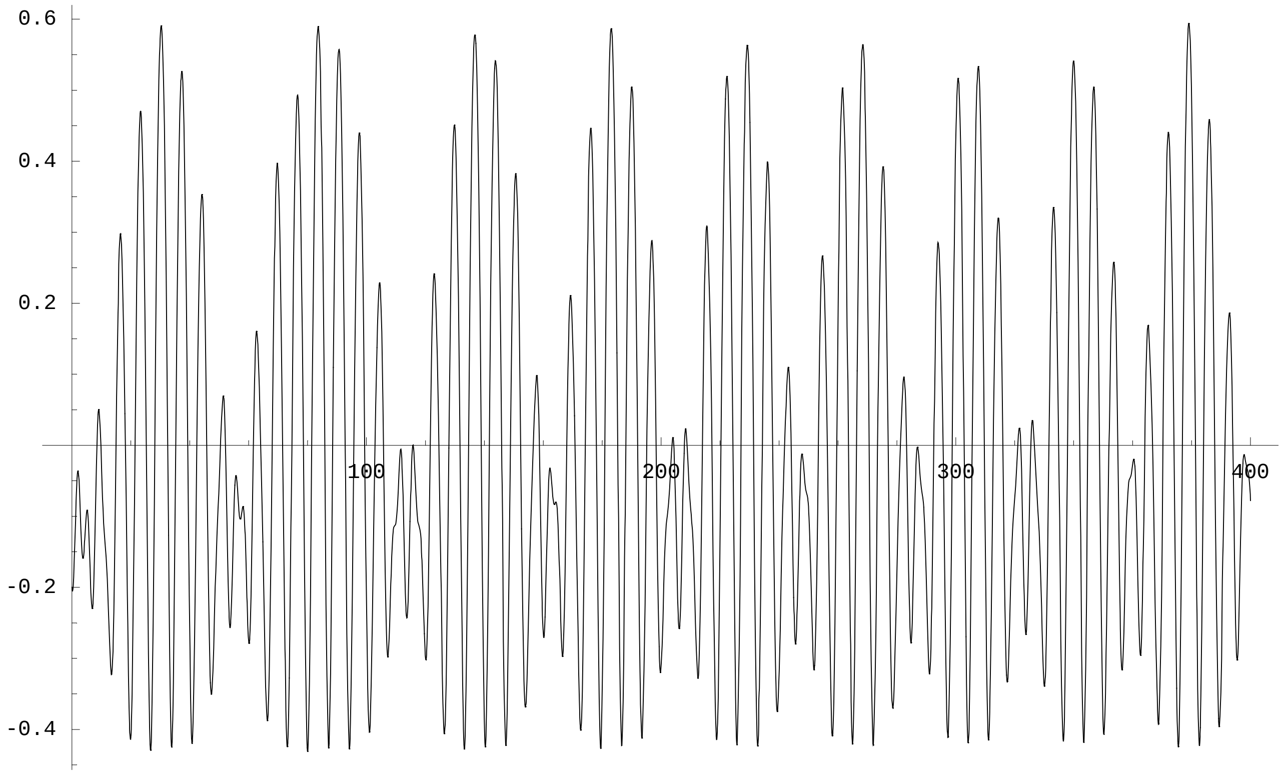

```
* Use Energy = 1 / 8 to find the initial value x'[0] : *

N[(.25 - (.05)^2 - (.1)^2 - (.2)^2 - 2 * (.1)^2 * (-0.2) + 2 * (-0.2)^3 / 3)^.5, 40]

0.442907

* For specified initial data,
solve coupled second - order differential equations for x[t] and y[t] : *
```

```
NDSolve[{x''[t] == -x[t] - 2 x[t] * y[t], y''[t] == -y[t] - x[t]^2 + y[t]^2, x[0] == -0.1,
  y[0] == -0.2, x'[0] == 0.4429070632386287`, y'[0] == -0.05}, {x, y}, {t, 0, 1000}]
```

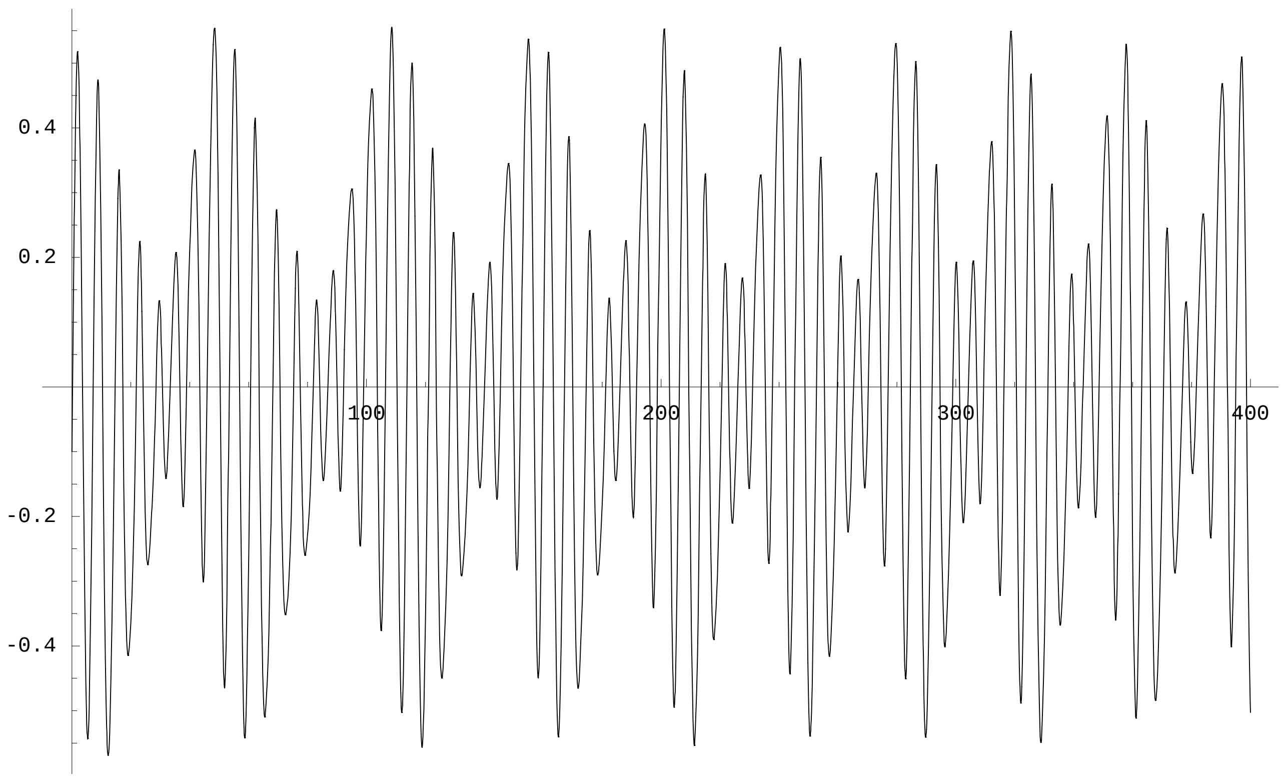
```
{{x → InterpolatingFunction[{{0., 1000.}}, <>],
  y → InterpolatingFunction[{{0., 1000.}}, <>]}}
```

*** Plot for t between 0 and 400. Note that motion is irregular : ***

```
Plot[Evaluate[y[t] /. %14], {t, 0, 400}, PlotPoints → 1000, ImageSize → {600, 400}];
```



```
Plot[Evaluate[x[t] /. %14], {t, 0, 400}, PlotPoints → 1000, ImageSize → {600, 400}];
```



*** Plot the Poincare section. Points do not lie on invariant curves (invariant tori have been destroyed) : ***

```
ParametricPlot3D[Evaluate[{x[t], y[t], y'[t]} /. %8], {t, 0, 1000}, PlotPoints → 4000,
  PlotRange → {{0, 0.001}, {-.6, .8}, {-.6, .6}}, ViewPoint -> {1, 0, 0},
  AxesLabel → {" ", "y", "y'"}, Ticks → {{ }, {-.3, 0, .3}, {-.3, 0, .3}},
  TextStyle → {FontFamily -> "Times", FontSize → 16}, ImageSize → {500, 500}];
```