Phys 106b
HW #3

① a. Note that, for $k\alpha$ = integer

$$\sum_{n=0}^{N-1} e^{2\pi i k\alpha n} = N, \text{ and for } k\alpha \neq \text{integer}$$

$$\left|\sum_{n=0}^{N-1} e^{2\pi i k\alpha n}\right| = \left|\frac{e^{2\pi i k N\alpha}-1}{e^{2\pi i k\alpha}-1}\right| \leq \frac{2}{|e^{2\pi i k\alpha}-1|}$$

Therefore $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k\alpha n} = 0$ for $k\alpha \neq$ int,

and $$\langle A\rangle_{time} = \lim_{N\to\infty} \frac{1}{N}\sum_n\sum_k \tilde{A}(k)\exp\left[2\pi i k(x_0+n\alpha)\right]$$

$$= \tilde{A}(k=0) + \sum_{k\neq 0}\tilde{A}(k)e^{2\pi i k x_0} \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i k\alpha n}$$

$$= \tilde{A}(k=0)$$

b. $$\langle A\rangle_{space} = \int_0^1 dx \sum_k \tilde{A}(k)e^{2\pi i k x} = \tilde{A}(k=0)$$

(2) $\tilde{M}(y) = g(M(g^{-1}(y)))$

$$\Rightarrow \tilde{M}'(y) = \frac{d}{dy}\tilde{M}(y) = g'((M(y)))M'(g^{-1}(y))(g^{-1})'(y)$$

(by chain rule)

$$= g'(M(x))\, M'(x) \left[g'(x)\right]^{-1}$$

(by inverse function theorem)

Now $h = \int_0^1 dx\, \rho(x) \ln|M'(x)|$

and $\tilde{h} = \int_0^1 dy\, \tilde{\rho}(y) \ln|\tilde{M}'(y)|$, but $dx\,\rho(x) = dy\,\tilde{\rho}(y)$

$$\Rightarrow \tilde{h} = h + \int dx\, \rho(x) \ln\left|\frac{g'(M(x))}{g'(x)}\right|$$

But recall $\rho(x)$ is invariant under the map M, which means $\int dx\, \rho(x)\left[f(M(x)) - f(x)\right] = 0$, (for any function f)

so in particular

$$\int dx\, \rho(x) \left[\ln|g'(M(x))| - \ln|g'(x)|\right] = 0$$

and therefore $\tilde{h} = h$

(3) $\begin{pmatrix} x \\ y \end{pmatrix} \to T\begin{pmatrix} x \\ y \end{pmatrix} \pmod 1$ $\qquad T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

- orbit of length 1 (fixed point)

$$x + y = x \pmod 1 \Rightarrow \quad y = \text{integer}$$
$$x + 2y = y \pmod 1 \Rightarrow \quad x = -y \pmod 1 = \text{integer}$$

therefore $\quad (x, y) = (0, 0) \pmod 1$

- Orbits of length 2 (fixed points of T^2)

$$T^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \Rightarrow \quad \begin{aligned} 2x + 3y &= x \pmod 1 \\ 3x + 5y &= y \pmod 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} x + 3y &= n \\ 3x + 4y &= m \end{aligned} \Rightarrow \quad \begin{aligned} 5y &= 3n - m = \text{integer} \\ 5x &= 3m - 4n = \text{integer} \end{aligned}$$

Aside from $(x, y) = (0, 0)$, the other solutions in $I \times I$ are

$\left(\frac{1}{5}, \frac{3}{5}\right) \quad (n = 2, m = 3)$

$\left(\frac{2}{5}, \frac{1}{5}\right) \quad (n = 1, m = 2)$

$\left(\frac{3}{5}, \frac{4}{5}\right) \quad (n = 3, m = 5)$

$\left(\frac{4}{5}, \frac{2}{5}\right) \quad (n = 2, m = 4)$

orbits are:

$$\left(\tfrac{1}{5}, \tfrac{3}{5}\right) \to \left(\tfrac{4}{5}, \tfrac{2}{5}\right) \to \left(\tfrac{1}{5}, \tfrac{3}{5}\right)$$
$$\left(\tfrac{2}{5}, \tfrac{1}{5}\right) \to \left(\tfrac{3}{5}, \tfrac{4}{5}\right) \to \left(\tfrac{2}{5}, \tfrac{1}{5}\right)$$

- Orbits of length 3 (fixed points of T^3)

$$T^3 = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \Rightarrow \quad \begin{aligned} 5x + 8y &= x \pmod 1 \\ 8x + 13y &= y \pmod 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} 4x + 8y &= n \\ 8x + 12y &= m \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4y &= 2n - m \\ 4x &= 2m - n \end{aligned}$$

therefore $4x$ and $4y$ must be integers – and any (x, y) where $4x$ and $4y$ are integers is a solution:

$x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$
$y = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ $\Rightarrow$ excluding $(x, y) = (0, 0)$, there are $4^2 - 1 = 15$ solutions, and therefore $15/3 = 5$ orbits of length 3.

4a) $M(x) = rx - rx^2$ If $M(x) = x$, then

$0 = (r-1)x - rx^2 \Rightarrow$ either $x = 0$ or $x = \frac{r-1}{r}$.

Therefore, for $x \in I = [0,1]$, fixed points are

$x_0 = 0$ for all r

$x_0 = \frac{r-1}{r}$ for $r \geq 1$

Stability: At $x_0 = 0$, $M'(x_0) = r$

$x_0 = \frac{r-1}{r}$, $M'(x_0) = r - 2rx_0 = 2 - r$

Therefore,

Fixed point at $x_0 = 0$ is $\begin{cases} \text{stable} & r < 1 \\ \text{unstable} & r > 1 \end{cases}$

Fixed point at $x_0 = \frac{r-1}{r}$ is $\begin{cases} \text{stable} & 1 < r < 3 \\ \text{unstable} & r > 3 \end{cases}$

b) $M^2(x) = M(M(x)) = r[rx(1-x)][1 - rx(1-x)]$

$= -xr^2(rx^3 - 2rx^2 + (r+1)x - 1)$

Fixed points of $M^2(x)$: $M^2(x) = x \Rightarrow$ either $x = 0$

or $rx^3 - 2rx^2 + (r+1)x - 1 + \frac{1}{r^2} = 0$

One solution is $x_0 = \frac{r-1}{r}$, as in (a) above

The other 2, for 3.3 are $x_1 = .4794$

$x_2 = .8236$

(found with Mathematica, or you could divide out $x - \frac{r-1}{r}$, to obtain a quadratic equation)

Stability coefficient is

$M'(x_1)M'(x_2) = r^2(1 - 2x_1)(1 - 2x_2) = -.2900$