Physics 106b

Problem Set 2

(1) a. Representing a number by its binary expansion, we see that a period-4 point must have the form $.b_1 b_2 b_3 b_4 b_1 b_2 b_3 b_4 \cdots$, so there are only $2^4 = 16$ possibilities. Of these, two represent the same period-1 orbit ($.0000\cdots = 0 \equiv .1111\cdots = 1 \pmod 1$), and two are in a period-2 orbit ($.0101\cdots = \frac{1}{3}$, $.1010\cdots = \frac{2}{3}$). The remaining 12 are in three period-4 orbits:

$$.0001\cdots = \tfrac{1}{15} \to .0010\cdots = \tfrac{2}{15} \to .0100\cdots = \tfrac{4}{15} \to .1000\cdots = \tfrac{8}{15} \to \tfrac{1}{15}$$

$$.0011\cdots = \tfrac{1}{5} \to .0110\cdots = \tfrac{2}{5} \to .1100\cdots = \tfrac{4}{5} \to .1001\cdots = \tfrac{3}{5} \to \tfrac{1}{5}$$

$$.0111\cdots = \tfrac{7}{15} \to .1110\cdots = \tfrac{14}{15} \to .1101\cdots = \tfrac{13}{15} \to .1011\cdots = \tfrac{11}{15} \to \tfrac{7}{15}$$

So there are 3 orbits of period 4.

b. There are 2^p points of the form $.b_1 \cdots b_p b_1 \cdots b_p \cdots$. These points must have period dividing p, so each has period 1 or p. Two represent the same period-1 orbit as in (a); the remainder must all have period p. Each orbit contains exactly p points, so there are

$$\frac{2^p - 2}{p} \text{ orbits of period } p.$$

(This is always an integer, a consequence of Fermat's Theorem: For p prime and a any integer, p divides $(a^p - a)$ exactly.)

If p is not prime, some orbits will have periods which divide p (as seen in (a)), which makes the formula more complicated.

FOR YOUR AMUSEMENT:

[The general formula is $N(n) = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right) 2^k$, where the sum is taken over all (positive) numbers k dividing n, $N(n)$ is the number of orbits of period n, and

$$\mu(n) = \begin{cases} 0 & \text{if a perfect square (greater than 1) is a divisor of } n \\ +1 & \text{if } n \text{ has an even number of distinct prime factors} \\ -1 & \text{if } n \text{ has an odd number of distinct prime factors} \end{cases}$$

(for ± 1: and no perfect square greater than 1 divides n.)

Then $N(p) = \frac{1}{p}\left[\mu(1) 2^p + \mu(p) 2^1\right] = \frac{1}{p}(2^p - 2)$ as above.]

② a. Length of orbit is smallest positive integer n such that $x + n\alpha = x + \text{integer}$, or $np/q = \text{integer}$. This is $n = q$ if p and q have no common factor

b. If $M^n(x) = M^m(x)$, then

$$\text{integer} = n\alpha - m\alpha = (n-m)\alpha \Rightarrow \alpha = \frac{\text{integer}}{n-m} = \text{rational}$$

c. From (b), the set $\{\alpha n \ (\text{mod } 1),\ n = 0, 1, 2, 3, \dots\}$ is an infinite set in I; therefore for any $\varepsilon > 0$ there are integers n, m $(n > m \geq 0)$ such that

$$\varepsilon > |n\alpha - m\alpha \ (\text{mod } 1)|$$
$$= |K\alpha \ (\text{mod } 1)|, \text{ where } K = n - m$$

Now, consider the orbit of M^K $\{x, M^K(x), M^{2K}(x), \dots\}$

The successive points are separated by distance $< \varepsilon$ and wind around the circle; therefore, for any y, y is within distance ε of some point on the orbit.

(3) Let $.b$ be an n-bit approximation to $x \in [0,1]$
$.c$ be an m-bit approximation to $y \in [0,1]$.

Let c' be c in reverse order
b' be b in reverse order

Then $\ldots b c' b c' . b c' b c' \ldots$

is a periodic point with period $n+m$; for this point $x' = .bc'bc'bc'\ldots$
$y' = .cb'cb'cb'$

Since n and m can be arbitrarily large, (x', y') can be arbitrarily close to (x,y).

(4)

In[1]:=

```
M[x_] := 4*x*(1-x)
```

In[6]:=

```
N[NestList[M,N[1/10+10^(-17),40],50]-NestList[M, N[1/10,40],50]]
```

↑ be sure to specify enough precision!

Out[6]=

{1. 10^{-17}, 3.2 10^{-17}, 3.584 10^{-17}, -1.20881 10^{-16},
-2.04034 10^{-16}, 5.25493 10^{-16}, -3.59103 10^{-16},
1.35256 10^{-15}, 4.18387 10^{-15}, 3.28103 10^{-15},
-1.21152 10^{-14}, -3.41323 10^{-14}, 1.07138 10^{-15},
-4.28501 10^{-15}, -1.71316 10^{-14}, -6.83914 10^{-14},
-2.71412 10^{-13}, -1.0516 10^{-12}, -3.68694 10^{-12},
-7.91278 10^{-12}, 1.34279 10^{-11}, -3.4377 10^{-11},
2.48517 10^{-11}, -9.29128 10^{-11}, -2.77709 10^{-10},
-1.29645 10^{-10}, 5.04452 10^{-10}, 1.80093 10^{-9}, 4.27302 10^{-9},
-5.06438 10^{-9}, 1.67006 10^{-8}, 2.4003 10^{-8}, -7.12204 10^{-8},
-2.86282 10^{-8}, 1.122 10^{-7}, 4.12908 10^{-7}, 1.14442 10^{-6},
-1.8207 10^{-7}, 7.25976 10^{-7}, 2.86721 10^{-6}, 0.0000108929,
0.0000350382, 0.0000411111, -0.000136146, -0.000201977,
0.000585645, 0.000119312, -0.000474771, -0.00185977,
-0.00682947, -0.0187304} ← $M^{50}(.1+10^{-17}) - M^{50}(.1)$

In[7]:=

```
Fit[Log[Abs[%6]],{1,n},n]
```

Out[7]=

-40.1458 + 0.694732 n ← h_{approx}

In[8]:=

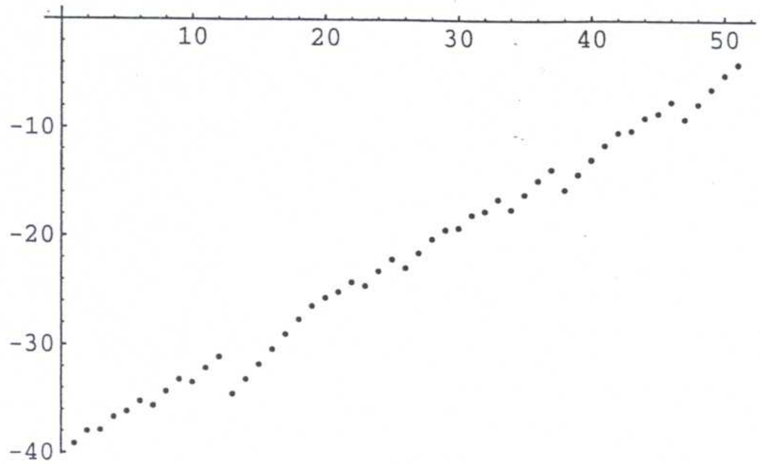
```
N[Log[2]]
```

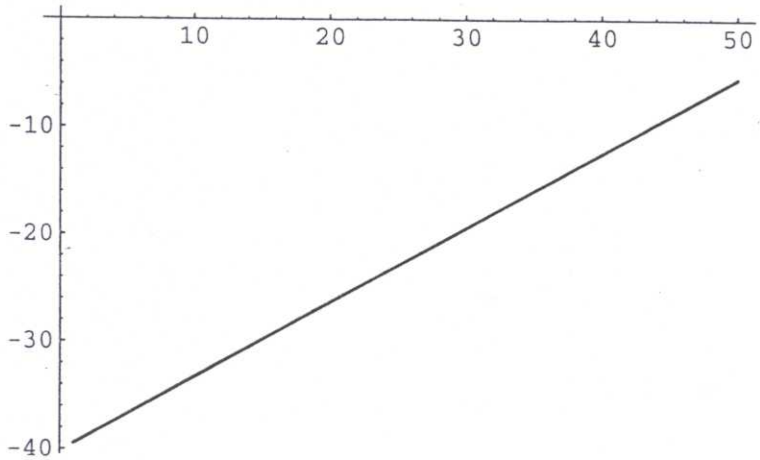
Out[8]=

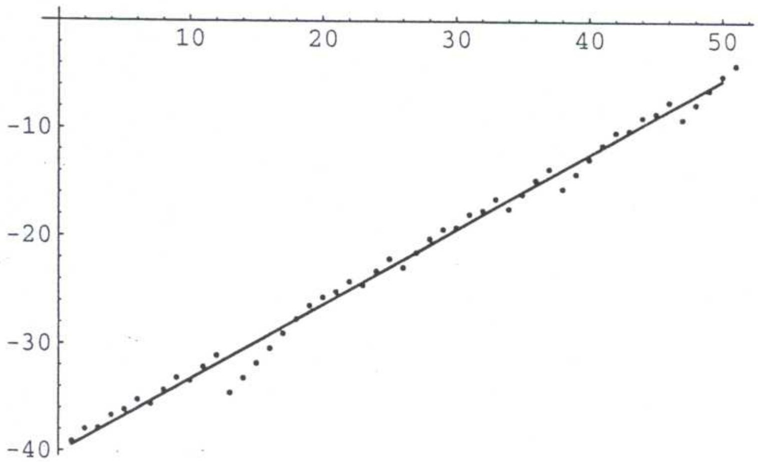
0.693147 ← h_{exact}

In[10]:=

```
Show[ListPlot[Log[Abs[%6]]],Plot[%7,{n,1,50}]];
```


10
20
30
40
50
-10
-20
-30
-40


10
20
30
40
50
-10
-20
-30
-40


10
20
30
40
50
-10
-20
-30
-40