Physics 106b
Solution Set #1

1. $$\hat H = \frac{1}{2m}\hat p^2 + \frac{c}{2}(\hat q \hat p + \hat p \hat q) = \frac{1}{2m}\hat p^2 + \frac{c}{2}(2\hat q\hat p + [\hat p, \hat q]) =$$
$$= \frac{1}{2m}\hat p^2 + c\hat q\hat p - \frac{i\hbar c}{2}$$

$$\langle q_j t_j | p_{j-1} t_{j-1}\rangle = \langle q_j t_{j-1}| e^{-i\hat H \Delta t/\hbar} | p_{j-1} t_{j-1}\rangle =$$
$$= \langle q_j t_{j-1}| 1 - i\Delta t\left[\frac{1}{2m}\hat p^2 + c\hat q\hat p - \frac{i\hbar c}{2}\right]/\hbar + O(\Delta t^2)| p_{j-1}t_{j-1}\rangle =$$
$$= \left(1 - \frac{i\Delta t}{\hbar}\left[\frac{1}{2m}p_{j-1}^2 + c q_j p_{j-1} - \frac{i\hbar c}{2}\right] + O(\Delta t^2)\right)\langle q_j t_{j-1}|p_{j-1} t_{j-1}\rangle =$$
$$= e^{-\frac{i\Delta t}{\hbar}H(q_j,p_{j-1})}\cdot e^{-\frac{c}{2}\Delta t}\cdot e^{-\frac{i}{\hbar}q_j p_{j-1}} + O(\Delta t^2)$$

So $$\underline{\langle q' t' | q t\rangle} = \lim_{\Delta t\to 0}\int \prod_j \left(\frac{dq_j\, dp_j}{2\pi\hbar}\right)\frac{dp_0}{2\pi\hbar}\exp\left[\sum_j \frac{i}{\hbar}(p_j \dot q_j - H(q_j,p_j))\Delta t - \frac{c}{2}\Delta t\right] = \underline{e^{-\frac12 c(t'-t)}\int dp\,dq\, e^{\frac{i}{\hbar}S}},$$
$$\underline{S = \int dt\,(p\dot q - H(q,p))}$$

2. $$p_i\dot q_i - H(q_i,p_i) = -\frac{1}{2m}p_i^2 + p_i(\dot q_i - c q_i) = -\frac{1}{2m}p_i'^2 + \frac{m}{2}(\dot q_i - c q_i)^2$$
where $p_i' = p_i - m(\dot q_i - c q)$, i.e.
$$\int \frac{dp_i}{2\pi\hbar} e^{+\frac{i}{\hbar}\left[-\frac{p_i'^2}{2m} + \frac{m}{2}(\dot q_i - c q_i)^2\right]\Delta t} = e^{\frac{i}{\hbar}\frac{m}{2}(\dot q_i - cq_i)^2\Delta t}\int\frac{dp_i'}{2\pi\hbar}e^{-\frac{i p_i'^2\Delta t}{2m\hbar}} =$$
$$= e^{\frac{i}{\hbar}\frac{m\Delta t}{2}(\dot q_i - c q_i)^2}\cdot\frac{1}{2\pi\hbar}\cdot\left(\frac{2\pi\hbar m}{i\Delta t}\right)^{\frac12}$$
so $$\langle q't'|qt\rangle = e^{-\frac{c}{2}(t'-t)}\lim_{\Delta t\to0}\int\prod_j\left[\frac{dq_j}{(2\pi\hbar)^{\frac12}}\left(\frac{m}{i\Delta t}\right)^{\frac12}\right] e^{\frac{i}{\hbar}\sum_j \Delta t\,\frac{m}{2}(\dot q_j - c q_j)} =$$
$$= e^{-\frac{c}{2}(t'-t)}\int (dq)'\, e^{i\frac{S[q]}{\hbar}},$$ where $S[q] = \int \frac{m(\dot q - cq)}{2}d$