
Consider a particle moving in two dimensions, governed by the Hamiltonian

\[ H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \]

where \( V(x, y) \) is the Hénon-Heiles potential

\[ V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3. \]

a) Numerically integrate the equations of motion on the energy surface \( H = E = 1/12 \), with initial data

\[
\begin{align*}
  x(0) &= -0.1, \\
  y(0) &= -0.2, \\
  p_y(0) &= -0.05.
\end{align*}
\]

Plot \( x(t) \) and \( y(t) \) for time \( t \) in the interval \((0, 400)\).

Hint: Use the software of your choice. If you use Mathematica, the NDSolve command creates interpolating functions \( x[t] \) and \( y[t] \), which can be plotted using \( \text{Plot}[\text{Evaluate}[x[t]]] \) and \( \text{Plot}[\text{Evaluate}[y[t]]] \). It is recommended that you use Mathematica version 5.2 rather than version 6.

b) Plot the Poincaré section of your solution from (a) on the two-dimensional slice through the energy surface at \( x = 0 \), showing points where the trajectory passes through the slice for time \( t \) in the interval \((0, 1000)\). Choose the coordinates \( (y, p_y) \) on the slice.

Hint: If you use Mathematica, the ParametricPlot3D command can plot the points in a narrow slice near the \( x = 0 \) surface.


The same as problem (1), but now with energy \( E = 1/8 \).
3. Elliptic and hyperbolic fixed points of a two-dimensional map.

Consider the two-dimensional map

\[ M : \begin{cases} x \rightarrow x' = x + y, \\ y \rightarrow y' = y + f(x + y), \end{cases} \]

where \( x \) and \( y \) are real numbers. Here, \( f \) is a differentiable function, its only zero is at the origin, \( f(0) = 0 \), and its derivative at the origin is \( f'(0) = K \).

(a) Express the \( 2 \times 2 \) first derivative matrix of \( M \) in terms of the derivative of \( f \). Is the map \( M \) area preserving?

(b) Find the unique fixed point of \( M \). For what values of \( K \) is the fixed point elliptic? For what values of \( K \) is it hyperbolic?

4. Poincaré-Cartan theorem.

The Poincaré-Cartan theorem (see page 212 of Ott) asserts that for two closed curves \( \Gamma_1 \) and \( \Gamma_2 \) that enclose the same “tube” of trajectories in \((2N+1)\)-dimensional extended phase space,

\[ \oint_{\Gamma_1} \omega = \oint_{\Gamma_2} \omega, \]

where \( \omega \) is the one-form

\[ \omega = p_i dq_i - H dt. \]

To prove the theorem, we use Stokes’ theorem:

\[ \oint_{\Gamma_1} \omega - \oint_{\Gamma_2} \omega = \int_{\Sigma} d\omega, \]

where \( \Sigma \) is a two-surface along the tube with boundary \( \partial \Sigma = \Gamma_1 - \Gamma_2 \), and

\[ d\omega = dp_i \wedge dq_i - dH \wedge dt. \]

It remains to show that the integral over \( \Sigma \) vanishes. For this purpose, we parametrize \( \Sigma \) with variables \((s, t)\), where \( s \) labels a trajectory in the tube, and \( t \) is the time along the trajectory, and then “pull back” the two-form \( d\omega \) to the \((s, t)\) space. Writing

\[
\begin{align*}
dp_i &= \frac{\partial p_i}{\partial s} ds + \frac{\partial p_i}{\partial t} dt \\
dq_i &= \frac{\partial q_i}{\partial s} ds + \frac{\partial q_i}{\partial t} dt \\
dH &= \frac{\partial H}{\partial p_i} dp_i ds + \frac{\partial H}{\partial q_i} dq_i ds + \frac{\partial H}{\partial p_i} dp_i dt + \frac{\partial H}{\partial q_i} dq_i dt
\end{align*}
\]

and using Hamilton’s equations, complete the proof of the Poincaré-Cartan theorem. (Recall that the wedge product is antisymmetric: \( ds \wedge ds = 0 = dt \wedge dt \) and \( ds \wedge dt = -dt \wedge ds \).)