Ph 106b

Homework Assignment No. 4
Due: Thursday, 7 February 2008

1. Regular motion in the Hénon-Heiles Potential.

Consider a particle moving in two dimensions, governed by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) ,$$

where $V(x, y)$ is the *Hénon-Heiles potential*

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3 .$$

a) Numerically integrate the equations of motion on the energy surface $H = E = 1/12$, with initial data

$$\begin{aligned} x(0) &= -0.1 , \\ y(0) &= -0.2 , \\ p_y(0) &= -0.05 . \end{aligned}$$

Plot $x(t)$ and $y(t)$ for time t in the interval $(0, 400)$.

Hint: Use the software of your choice. If you use *Mathematica*, the NDSolve command creates interpolating functions x[t] and y[t], which can be plotted using Plot[Evaluate[x[t]]] and Plot[Evaluate[y[t]]]. It is recommended that you use *Mathematica* version 5.2 rather than version 6.

b) Plot the Poincaré section of your solution from (*a*) on the two-dimensional slice through the energy surface at $x = 0$, showing points where the trajectory passes through the slice for time t in the interval $(0, 1000)$. Choose the coordinates (y, p_y) on the slice.

Hint: If you use *Mathematica*, the ParametricPlot3D command can plot the points in a narrow slice near the $x = 0$ surface.

2. Irregular motion in the Hénon-Heiles Potential.

The same as problem (1), but now with energy $E = 1/8$.

3. Elliptic and hyperbolic fixed points of a two-dimensional map.

Consider the two-dimensional map

$$M : \begin{cases} x \to x' = x + y \ , \\ y \to y' = y + f(x+y) \ , \end{cases}$$

where x and y are real numbers. Here, f is a differentiable function, its only zero is at the origin, $f(0)$=0, and its derivative at the origin is $f'(0) = K$.

a) Express the 2×2 first derivative matrix of M in terms of the derivative of f. Is the map M area preserving?

b) Find the unique fixed point of M. For what values of K is the fixed point elliptic? For what values of K is it hyperbolic?

4. Poincaré-Cartan theorem.

The Poincaré-Cartan theorem (see page 212 of Ott) asserts that for two closed curves Γ_1 and Γ_2 that enclose the same "tube" of trajectories in $(2N+1)$-dimensional extended phase space,

$$\oint_{\Gamma_1} \omega = \oint_{\Gamma_2} \omega \ ,$$

where ω is the one-form

$$\omega = p_i dq_i - H dt \ .$$

To prove the theorem, we use Stokes' theorem:

$$\oint_{\Gamma_1} \omega - \oint_{\Gamma_2} \omega = \int_{\Sigma} d\omega \ ,$$

where Σ is a two-surface along the tube with boundary $\partial\Sigma = \Gamma_1 - \Gamma_2$, and

$$d\omega = dp_i \wedge dq_i - dH \wedge dt \ .$$

It remains to show that the integral over Σ vanishes. For this purpose, we parametrize Σ with variables (s,t), where s labels a trajectory in the tube, and t is the time along the trajectory, and then "pull back" the two-form $d\omega$ to the (s,t) space. Writing

$$\begin{aligned} dp_i &= \frac{\partial p_i}{\partial s} ds + \frac{\partial p_i}{\partial t} dt \\ dq_i &= \frac{\partial q_i}{\partial s} ds + \frac{\partial q_i}{\partial t} dt \\ dH &= \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial s} ds + \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial s} ds + \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial t} dt + \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial t} dt \end{aligned}$$

and using Hamilton's equations, complete the proof of the Poincaré-Cartan theorem. (Recall that the wedge product is antisymmetric: $ds \wedge ds = 0 = dt \wedge dt$ and $ds \wedge dt = -dt \wedge ds$.)