1. An ergodic map that is not chaotic, part II. Consider the invertible map on the unit interval

\[ M(x) = x + \alpha \pmod{1} \equiv \begin{cases} x + \alpha , & \text{if } x + \alpha < 1 , \\ x + \alpha - 1 , & \text{if } x + \alpha \geq 1 . \end{cases} \]

where \( \alpha \in (0, 1) \) is an irrational real number. This map is continuous if we regard it as a map from the circle to the circle, by identifying the points 0 and 1. (It just rigidly rotates the circle by the angle \( 2\pi \alpha \) radians.) In this exercise, you will show that \( M \) is ergodic, with a uniform invariant density.

Let \( A(x) \) be a smooth function on the circle, or equivalently a periodic function of \( x \) with period 1. Then ergodicity means that a time average over a long orbit can be replaced by an integral over phase space:

\[ \langle A \rangle_{\text{time}} = \langle A \rangle_{\text{space}}, \]

where

\[ \langle A \rangle_{\text{time}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} A(M^n(x_0)), \]

\[ \langle A \rangle_{\text{space}} = \int_0^1 dx \ A(x), \]

Note that \( A(x) \), being periodic, can be expressed as a Fourier series

\[ A(x) = \sum_{k=-\infty}^{\infty} \tilde{A}_k e^{2\pi ikx}. \]

When we say the map is “smooth” we mean that the Fourier coefficients \( \tilde{A}_k \) decrease rapidly for large \( k \). For this problem, you may assume that the sum is truncated to a sum over a finite number of Fourier modes,

\[ A(x) = \sum_{k=-k_{\max}}^{k_{\max}} \tilde{A}_k e^{2\pi ikx}. \]

a) Evaluate the sum over \( n \) and take the limit \( N \to \infty \) to express \( \langle A \rangle_{\text{time}} \) in terms of the \( \tilde{A}_k \)’s.
b) Evaluate the integral to express $\langle A \rangle_{\text{space}}$ in terms of the $\tilde{A}_k$’s. Verify that $\langle A \rangle_{\text{time}} = \langle A \rangle_{\text{space}}$.

2. **Conjugate maps.** Recall that if $M$ is an *ergodic* differentiable one-dimensional map defined on the unit interval $I = [0, 1]$, the Lyapunov exponent $h$ of $M$ can be expressed as

$$h = \int_0^1 dx \rho(x) \ln|M'(x)|,$$

where $M'(x)$ denotes the first derivative of $M$ and $\rho(x)$ is a density function invariant under $M$. A map $\tilde{M}$ is said to be *conjugate* to $M$ if $\tilde{M}$ can be expressed as

$$\tilde{M} = g \circ M \circ g^{-1},$$

where $\circ$ denotes composition of mappings, and $g$ is a differentiable and invertible map from $I$ to $I$. Show that if $M$ and $\tilde{M}$ are conjugate they have the same Lyapunov exponent.

3. **Periodic orbits of the cat map.** Recall that the “cat map” $T$ is a chaotic map from the 2-torus to the 2-torus that is continuous, invertible, and area preserving. If we represent the 2-torus as a unit square with opposite sides identified, the action of $T$ on the point $(x, y) \in I \times I$ can be expressed as

$$T : \begin{cases} x \to x' = x + y \pmod{1}, \\ y \to y' = x + 2y \pmod{1}. \end{cases}$$

Find the periodic orbits of the cat map of length 1 and 2. How many periodic orbits are there of length 3?

4. **Stable orbits of a one-dimensional map.** Consider the logistic map on the unit interval

$$M(x) = rx(1 - x),$$

where $0 \leq r \leq 4$.

a) Find (analytically) all fixed points of $M$ in the unit interval, and for each fixed point, whether it is stable or unstable.

b) For $r = 3.3$, find the period-two orbit of $M$: $x_1 \to x_2 \to x_1$ (determine $x_1$ and $x_2$ numerically to four-digit accuracy). Find the stability coefficient $M'(x_1)M'(x_2)$ of this orbit (also to four-digit accuracy). **Hint:** It is useful to study the iterated map $M^2$. 