Ph 106b

Homework Assignment No. 3
Due: Thursday, 31 January 2008

1. **An ergodic map that is not chaotic, part II**. Consider the invertible map on the unit interval

$$M(x) = x + \alpha \pmod 1 \equiv \begin{cases} x + \alpha \ , & \text{if } x + \alpha < 1 \ , \\ x + \alpha - 1 \ , & \text{if } x + \alpha \geq 1 \ . \end{cases}$$

where $\alpha \in (0, 1)$ is an *irrational* real number. This map is continuous if we regard it as a map from the circle to the circle, by identifying the points 0 and 1. (It just rigidly rotates the circle by the angle $2\pi\alpha$ radians.) In this exercise, you will show that M is ergodic, with a uniform invariant density.

Let $A(x)$ be a smooth function on the circle, or equivalently a periodic function of x with period 1. Then ergodicity means that a time average over a long orbit can be replaced by an integral over phase space: $\langle A \rangle_{\text{time}} = \langle A \rangle_{\text{space}}$, where

$$\begin{aligned} \langle A \rangle_{\text{time}} &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} A(M^n(x_0)) \ , \\ \langle A \rangle_{\text{space}} &= \int_0^1 dx \ A(x) \ , \end{aligned}$$

Note that $A(x)$, being periodic, can be expressed as a Fourier series

$$A(x) = \sum_{k=-\infty}^{\infty} \tilde{A}_k e^{2\pi i k x} \ .$$

When we say the map is "smooth" we mean that the Fourier coefficients $\tilde{A}_k$ decrease rapidly for large k. For this problem, you may assume that the sum is truncated to a sum over a finite number of Fourier modes,

$$A(x) = \sum_{k=-k_{\max}}^{k_{\max}} \tilde{A}_k e^{2\pi i k x} \ .$$

a) Evaluate the sum over n and take the limit $N \to \infty$ to express $\langle A \rangle_{\text{time}}$ in terms of the $\tilde{A}_k$'s.

b) Evaluate the integral to express $\langle A \rangle_{\text{space}}$ in terms of the $\tilde{A}_k$'s. Verify that $\langle A \rangle_{\text{time}} = \langle A \rangle_{\text{space}}$.

2. Conjugate maps. Recall that if M is an *ergodic* differentiable one-dimensional map defined on the unit interval $I = [0, 1]$, the Lyapunov exponent h of M can be expressed as

$$h = \int_0^1 dx \rho(x) \ln |M'(x)| \ ,$$

where $M'(x)$ denotes the first derivative of M and $\rho(x)$ is a density function invariant under M. A map $\tilde{M}$ is said to be *conjugate* to M if $\tilde{M}$ can be expressed as

$$\tilde{M} = g \circ M \circ g^{-1} \ ,$$

where $\circ$ denotes composition of mappings, and g is a differentiable and invertible map from I to I. Show that if M and $\tilde{M}$ are conjugate they have the same Lyapunov exponent.

3. Periodic orbits of the cat map. Recall that the "cat map" T is a chaotic map from the 2-torus to the 2-torus that is continuous, invertible, and area preserving. If we represent the 2-torus as a unit square with opposite sides identified, the action of T on the point $(x, y) \in I \times I$ can be expressed as

$$T : \begin{cases} x \to x' = x + y \pmod 1 \ , \\ y \to y' = x + 2y \pmod 1 \ . \end{cases}$$

Find the periodic orbits of the cat map of length 1 and 2. How many periodic orbits are there of length 3?

4. Stable orbits of a one-dimensional map. Consider the logistic map on the unit interval

$$M(x) = rx(1 - x) \ ,$$

where $0 \leq r \leq 4$.

a) Find (analytically) all fixed points of M in the unit interval, and for each fixed point, whether it is stable or unstable.

b) For $r = 3.3$, find the period-two orbit of M: $x_1 \to x_2 \to x_1$ (determine x_1 and x_2 numerically to four-digit accuracy). Find the stability coefficient $M'(x_1)M'(x_2)$ of this orbit (also to four-digit accuracy). **Hint**: It is useful to study the iterated map M^2.