1. **Finite-length orbits of the shift map.** The “shift map” is a 2-to-1 map \( M \) on the unit interval \( I = [0, 1] \) defined by

\[
M(x) = \begin{cases} 
2x , & \text{if } x < \frac{1}{2} , \\
2x - 1 , & \text{if } x \geq \frac{1}{2} .
\end{cases}
\]

If \( x \) is expressed as a binary expansion \( x = .b_1b_2b_3b_4b_5\ldots \), i.e.,

\[
x = \sum_{n=1}^{\infty} b_n2^{-n} ,
\]

then the action of the shift map is

\[
M(.b_1b_2b_3b_4b_5\ldots) = .b_2b_3b_4b_5b_6\ldots ;
\]

the bit furthest to the left is erased, and all other bits are shifted one place to the left. (That’s why we call it the “shift map.”)

*a) If an orbit of a map returns to its starting point after \( n \) iterations of the map, and not before, we say that the orbit is of length \( n \). For the shift map, count the periodic orbits of length 4. (Don’t include the orbits of length 1 or 2, and don’t overcount—two orbits that sweep through the same points in the same cyclic order are considered to be the same orbit.)*

*b) For the shift map, how many orbits are there of length \( p \), where \( p \) is a prime number? (Why is it more difficult to answer this question in general, if \( p \) is not prime?)*

2. **An ergodic map that is not chaotic.** Consider the invertible map on the unit interval

\[
M(x) = x + \alpha \pmod{1} \equiv \begin{cases} 
x + \alpha , & \text{if } x + \alpha < 1 , \\
x + \alpha - 1 , & \text{if } x + \alpha \geq 1 .
\end{cases}
\]

where \( \alpha \in (0, 1) \) is a real number.
Suppose \( \alpha \) is a rational number, which can be expressed as \( \alpha = p/q \), where \( p \) and \( q \) are positive integers with no common factor. For \( x \in I = [0, 1] \), what is the length of the orbit of \( x \) under the map \( M \)?

b) Now suppose that \( \alpha \) is irrational. Show that for any \( x \in I \) and for any two distinct nonnegative integers \( n \) and \( m \), \( M^n(x) \neq M^m(x) \), where \( M^n \) denotes the \( n \)th iteration of the map \( m \) (and \( M^0 \) is the identity map).

c) For irrational \( \alpha \), show that for any \( x \in [0, 1] \), the orbit \( \{ M^n(x), n = 0, 1, 2, 3 \ldots \} \) is dense in \( I \). That is, show that for any positive \( \varepsilon \) and for any \( y \in I \), there exists an \( n \) such that \( |M^n(x) - y| < \varepsilon \). (Use the Bolzano-Weierstrass Theorem, which says that any infinite set in \( I \) has a point of accumulation.)

3. Periodic points of the baker’s map. The “baker’s map” is an invertible map \( M \) on \( I \times I \) defined by

\[
M(x, y) = \begin{cases} 
(2x, y/2), & \text{if } x < 1/2, \\
(2x - 1, y/2 + 1/2), & \text{if } x \geq 1/2.
\end{cases}
\]

If the binary expansions of \( x \) and \( y \) are \( x = .b_1b_2b_3b_4b_5 \ldots \) and \( y = .c_1c_2c_3c_4c_5 \ldots \), we can represent the point \((x, y)\) as two “back-to-back” sequences \( \ldots c_5c_4c_3c_2c_1, b_1b_2b_3b_4b_5 \ldots \). Then the action of \( M \) can be expressed as

\[
M : \ldots c_5c_4c_3c_2c_1, b_1b_2b_3b_4b_5 \ldots \to \ldots c_4c_3c_2c_1, b_1b_2b_3b_4b_5 \ldots ;
\]

each bit shifts one place to the left. We say that the point \( P = (x, y) \) is a periodic point of the map \( M \) if \( M^n(P) = P \) for some \( n \), where \( M^n \) denotes the \( n \)th iteration of the map. For the baker’s map, show that the periodic points are dense. That is, show that there is a periodic point of the map that is arbitrarily close to any point in the unit square \( I \times I \).

4. Numerical evaluation of a Lyapunov exponent. The Lyapunov exponent \( h \) of a one-dimensional map \( M \) is defined by

\[
M^n(x_0 + \epsilon) - M^n(x_0) \simeq C \epsilon^h \epsilon,
\]

for \( \epsilon \) small and \( n \) large (where \( C \) is a constant). Consider the \( r = 4 \) logistic map

\[
M(x) = 4x(1 - x)
\]

defined on the unit interval \( I \).
a) Numerically calculate

\[ M^n(.1 + 10^{-17}) - M^n(.1) \]

for \( n = 1, 2, 3, \ldots, 50 \). How far apart are the images of .1 and .1 + 10^{-17} after 50 iterations of \( M \)? (Be sure to specify enough precision in your computations!)

b) Estimate the Lyapunov exponent by fitting the data from (a) to the form \( Ce^{hn}e \). Compare to the exact value of \( h = \ln 2 \) derived in class.