Physics 106b
Midterm Solutions

J. Preskill

1 (a) Lyapunov exponent (10 points)

~~Separation between x and $x+\Delta x$ grows as~~

$$\Delta x \xrightarrow{M^n} a^n \Delta x = e^{hn} \Delta x \Rightarrow \boxed{h = \ln(a)}$$

(b) Orbits of prime order (10 points)

We represent M as a "shift map"

$$M: \ .b_1 b_2 b_3 b_4 \ldots \longrightarrow \ .b_2 b_3 b_4 b_5 \ldots ,$$

where $.b_1 b_2 b_3 b_4 \ldots$ is expansion in "base a":

$$x = \sum_{n=1}^{\infty} b_n a^{-n} \qquad b_n \in \{0, 1, 2, \ldots a-1\}$$

Fixed points of M^p are of the form $x = .b_1 b_2 \ldots b_p b_1 b_2 \ldots b_p$ —
There are a^p such points, of which a are fixed points of M $(b_1 = b_2 = \ldots = b_p)$ — the rest are periodic points of order p, and each orbit contains p such points. Therefore

$$\boxed{\text{No of orbits of length } p = \frac{a^p - a}{p}}$$

(c) Yes, periodic points are dense. Any real number x in the unit interval has expansion

$$x = .b_1 b_2 b_3 b_4 \ldots$$

Let b denote the string $b_1 b_2 b_3 \ldots b_n$

Then $y = .bbbb\ldots$ is periodic, with period that divides n, and $|x-y| < a^{-n}$,
since x and y agree to the first n places.

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2 (a) Periodic Orbits (10 points):

$J_0 = \text{rational} = \frac{r}{s}$ (r, s relatively prime integers)
w_0 may take any value in $[0, 1)$.

Orbit is $J = J_0 = \text{constant}$

$$w = w_0,\ w_0 + \frac{r}{s},\ w_0 + \frac{2r}{s},\ \cdots\ w_0 + \frac{(s-1)r}{s},\ w_0 + r \pmod 1$$

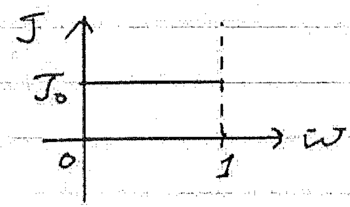
This is a periodic orbit of length r.

(w, J) is a periodic point for any rational J, and rationals are dense in real nos., so periodic points __are__ dense.

(b) Nonperiodic orbits (10 points):

orbit is nonperiodic for $J_0 = \text{irrational}$

orbit contains the points $w = w_0 + n J_0 \pmod 1$, $n \in \mathbb{Z}$
$J = J_0$



The orbit densely fills the "circle" (w, J_0), $w \in [0, 1)$

Time Average (10 points)

(c) The map is ergodic on the circle, with uniform invariant measure. Therefore

$$\langle J^2 \sin^2(\pi w) \rangle_{time} = \langle J^2 \sin^2(\pi w) \rangle_{space} = J_0^2 \int_0^1 dw \sin^2(\pi w) = \frac{1}{2} J_0^2$$

3 (a) Fixed points of new map (5 points):

$$w_{n+1} = w_n + J_n = w_n \pmod 1) \Rightarrow \boxed{J = \text{integer}}$$

$$J_{n+1} = J_n + \frac{F}{2\pi}\sin(2\pi w_{n+1})$$

$$\Rightarrow \sin(2\pi w_{n+1}) = 0 \Rightarrow \quad w = \tfrac{1}{2} \times (\text{integer})$$

or $\boxed{w = 0, \tfrac{1}{2}}$ in $[0, 1)$

(b) Linearized map (10 points):

$$\underset{\sim}{A} = \begin{bmatrix} \partial w_{n+1}/\partial w_n & \partial w_{n+1}/\partial J_n \\ \partial J_{n+1}/\partial w_n & \partial J_{n+1}/\partial J_n \end{bmatrix}$$

$$\frac{\partial w_{n+1}}{\partial w_n} = 1 = \frac{\partial w_{n+1}}{\partial J_n}$$

$$\frac{\partial J_{n+1}}{\partial w_n} = \frac{\partial w_{n+1}}{\partial w_n} F\cos(2\pi w_{n+1}) = F\cos(2\pi w_{n+1})$$

$$\frac{\partial J_{n+1}}{\partial J_n} = 1 + \frac{\partial w_{n+1}}{\partial J_n} F\cos(2\pi w_{n+1}) = 1 + F\cos(2\pi w_{n+1})$$

So --

$$\boxed{\underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ F\cos(2\pi w_{n+1}) & 1 + F\cos(2\pi w_{n+1}) \end{bmatrix}}$$

(c) Area-preserving (5 points)

$$\det \underset{\sim}{A} = 1 + F\cos(2\pi w_{n+1}) - F\cos(2\pi w_{n+1})$$

$$= 1 \Rightarrow \text{area-preserving}$$

3 (d) Elliptic and Hyperbolic Fixed Points (10 points):

From (c), two fixed points with $-\frac{1}{2} < J < \frac{1}{2}$

(i) $\bar{w} = 0,\ J = 0 \Rightarrow$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ F & 1+F \end{bmatrix}$$ Find eigenvalues of $\underset{\sim}{A}$

$$0 = \det(\lambda \underset{\sim}{\mathbb{I}} - \underset{\sim}{A}) = (\lambda - 1)(\lambda - 1 - F) - F = \lambda^2 - (2+F)\lambda + 1$$

Solutions are $\lambda_{\pm} = \frac{1}{2}\left(2 + F \pm \sqrt{(2+F)^2 - 4}\right)$

$$= 1 + \frac{F}{2} \pm \frac{1}{2}\sqrt{F(F+4)}$$

For $F > 0$, we have $\lambda_+ > 1 \Rightarrow$

$\boxed{(\bar{w} = 0,\ J = 0) \text{ is a hyperbolic fixed point}}$

(ii) $\bar{w} = \frac{1}{2},\ J = 0 \Rightarrow \underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ -F & 1-F \end{bmatrix}$

Differs from case (i) by $F \to -F$; thus,

$$\lambda_{\pm} = 1 - \frac{F}{2} \pm \frac{1}{2}\sqrt{F(F-4)}$$

λ_+ real (and $\lambda_+ > 1$) for $F > 4$

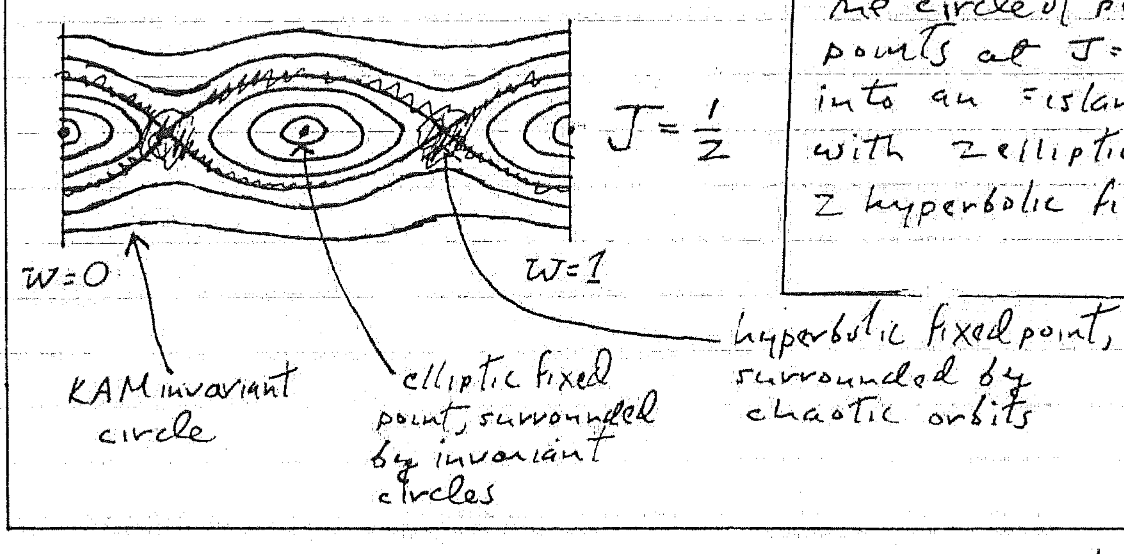
$\mathrm{Im}(\lambda_{\pm}) \neq 0$ for $0 < F < 4$

$(\bar{w} = \frac{1}{2},\ J = 0)$ is ... elliptic for $0 < F < 4$
hyperbolic for $F > 4$

3 (e) Fate of Period-2 orbits (1.0 points)

According to the KAM theorem, the invariant circles that are not "too close to rational" survive when F is small. But, according to the Poincare-Birkhoff theorem, the rational circles "break up" for $F > 0$.

Furthermore, the Poincare-Birkhoff theorem says that when the circle of period s is perturbed, $2Ks$ points of period s will survive (K = integer), of which Ks are elliptic fixed points of $(\text{Map})^s$ and Ks are hyperbolic fixed points, with the type of fixed point alternating as we travel along the circle. Typically, $K = 1$



The circle of period-2 points at $J = \frac{1}{2}$ breaks into an "island chain" with 2 elliptic and 2 hyperbolic fixed points.

Orbits are chaotic near the hyperbolic fixed points. Elliptical fixed points are surrounded by irrational invariant circles. The rational circles around the elliptical fixed points, upon closer inspection, break into still smaller island chains, etc...