

Invariant Tori

Let's return to the theory of integrable dynamical systems, and tie up a loose end in our earlier discussion of action-angle variables. We need to prove the Theorem (due to Liouville):

Theorem

In $2N$ -dimensional phase space, consider N functions of the canonical variables

$$F_1(q, p), F_2(q, p) \dots, F_N(q, p)$$

(i) Suppose that these functions are independent (meaning that, at each point, the N Hamiltonian vector fields that they generate are linearly independent).

(ii) Suppose that these functions are "in involution" - the Poisson bracket of any pair vanishes,

$$[F_i, F_j] = 0.$$

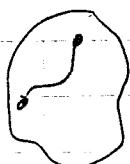
(iii) Consider the manifold $M(c_1, c_2, \dots, c_N)$ defined by specifying

$$F_1(q, p) = c_1$$

$$F_2(q, p) = c_2$$

$$\vdots$$

$$F_N(q, p) = c_N$$



Suppose that this manifold is compact (has finite volume) and connected (there is a continuous curve connecting any two points).

Then --

- $M(c_1, \dots, c_N)$ is (diffeomorphic to) a torus. That is, there is a differentiable 1-1 mapping from a product $S^1 \times \dots \times S^1$ of N circles to $M(c_1, \dots, c_N)$
- Thus, if a Hamiltonian system has N conserved quantities (one of which is H) fixing the values of these N quantities determines an N -torus in phase space. Any initial point on the torus remains on the torus under the Hamiltonian flow.
- Furthermore, a canonical transformation to action-angle variables can be constructed. This transformation can be reduced to quadrature — algebraic operations and the evaluation of ordinary integrals. Thus, a dynamical system with N conserved quantities is completely integrable.

Sketch of the Proof:

The idea of the theorem is that the manifold $M(c_1, \dots, c_N)$ can be generated by the flows associated with the N (Hamiltonian) vector fields generated by F_1, \dots, F_N , starting at an arbitrary point in M .



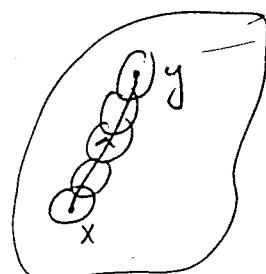
Because the N F_j 's are "in involution," the flow generated by each F_j preserves the values of all of the F_i 's, and hence preserves $M_{(c_1, \dots, c_N)}$. At each point in M , the N -independent vector fields generated by F_1, \dots, F_N preserve the N -dimensional tangent space of M at that point.



Now for each F_j define a canonical transformation $g_j(t_j)$ — the flow generated by "Hamiltonian" F_j , integrated for "time" t_j . Because the flows commute, $g_i(t_i)$ and $g_j(t_j)$ are commuting canonical transformations. Now consider the transformation

$$g(t_1, t_2, \dots, t_N) : x \mapsto g_1(t_1)g_2(t_2)\dots g_N(t_N)x$$

This transformation, for t_1, \dots, t_N infinitesimal acts transitively on the tangent space at x — i.e. it can take x to any nearby point in M , for suitable t_1, \dots, t_N . In fact



$g(t_1, \dots, t_N)$ acts transitively on all of M — for each x and y , there are values of the t_i such that

$$y = \vec{g}(\vec{t})x$$

This is because M is connected. There is a curve from x to y , and we can cover it with a finite number of open sets -- where flow acts transitively in each of the open sets --

$$\text{so } g(\vec{t}): x \rightarrow g_1(t_1) - g_n(t_n) x$$

is a mapping from \mathbb{R}^N onto M

But this mapping is not 1-1, since M is compact and \mathbb{R}^N is not. We want to find a subset of \mathbb{R} that is mapped 1-1 onto M .

Note that each Hamiltonian flow satisfies

$$g_j(t) g_j(s) = g_j(t+s),$$

$$\text{so we have } g(\vec{t}) g(\vec{s}) = g(\vec{s}) g(\vec{t}) = g(\vec{s} + \vec{t})$$

We can represent M as

$$M = \{ g(\vec{t}) x_0, \vec{t} \in \mathbb{R}^N \}$$

for some arbitrary x_0 in M -- we want to find all points in \mathbb{R}^N that are mapped to a single point in M

Defⁿ the stationary group $P(x_0)$ of \mathbb{R}^N is the set of (t_1, \dots, t_n) such that

$$g(\vec{t}) x_0 = x_0$$

- This is a group: $g(\vec{t}) x_0 = g(\vec{s}) x_0 \Rightarrow g(\vec{t} + \vec{s}) x_0 = x_0$

$$g(\vec{t})^{-1} = g(-\vec{t})$$



- It is the same group for each point x_0 :

Consider another point y_0 .

$y_0 = g(\vec{r})x_0$ for some $\vec{r} \in \mathbb{R}^N$, because mapping is transitive

$$g(\vec{t})y_0 = g(\vec{t})g(\vec{r})x_0 = g(\vec{r})g(\vec{t})x_0 = g(\vec{r})x_0 = y_0.$$

— if $g(\vec{t})x_0 = x_0$

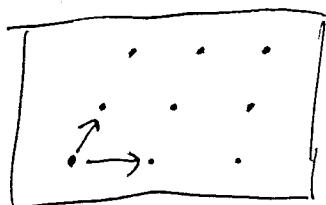
- It is discrete — since $g(\vec{t})x_0 \neq x_0$

for \vec{t} infinitesimal. So $g(\vec{t} + \vec{\alpha})x_0 \neq x_0$ if $\vec{\alpha} \in \Gamma$. (For each $\vec{t} \in \Gamma$, there is a neighborhood of \vec{t} that contains no other elements of Γ)

Lemma (an exercise)

If Γ is a discrete subgroup of \mathbb{R}^N (under addition), then there are K vectors (linearly independent) in \mathbb{R}^N such that

$$\Gamma = \left\{ n_1 \vec{v}_1 + n_2 \vec{v}_2 + \dots + n_K \vec{v}_K, n_1, n_2, \dots \in \mathbb{Z} \right\}$$



— a regular lattice in \mathbb{R}^N

Now, we have $K \leq N$ — so complete basis for \mathbb{R}^N to $\vec{v}_1, \dots, \vec{v}_N$

Now consider

$$x = g(w_1 \vec{v}_1) g(w_2 \vec{v}_2) \cdots g(w_N \vec{v}_N) x_0$$

We have $x = x'$ if and only if

$$w_1' = w_1 + \text{integer}$$

$$w_k' = w_k + \text{integer}$$

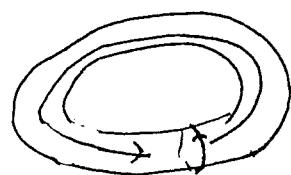
Since M is compact, we must have $k = N$, and

$$M = \left\{ x = g(w_1 \vec{v}_1) - g(w_N \vec{v}_N) x_0, \quad w_1, 2, \dots, N \in [0, 1] \right\}$$

This is a 1-1 differentiable map from $S^1 \times \cdots \times S^1$ onto M (The w 's are periodic variables with period 1)

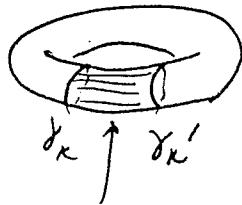
We have proved that the manifold determined by $F_1 = C_1, \dots, F_N = C_N$ is an N -torus. To complete the demonstration of integrability, we need to construct the canonical transformation to action-angle variables.

For fixed $F_1 = C_1, \dots, F_N = C_N$, we define N action variables J_1, \dots, J_N by selecting N independent "cycles" on the torus — i.e. the closed paths



$$J_i = \{g(w_i \vec{v}_i) x_0, \quad 0 \leq w_i \leq 1\}$$

Then $J_K(c_1, \dots, c_N) = \oint_{\gamma_K} p_i dq_i$



(Aside: action variable actually depends only on the homology of the cycle — i.e. two closed paths that form a boundary of a 2-dim region define the same J_K)

use Stokes theorem:

$$J_{K'} - J_K = (\oint_{\gamma_{K'}} - \oint_{\gamma_K}) p_i dq_i = \int_S dp_i \wedge dq_i$$

But the canonical two-form $dp_i \wedge dq_i$ vanishes on the torus. Consider two tangent vectors

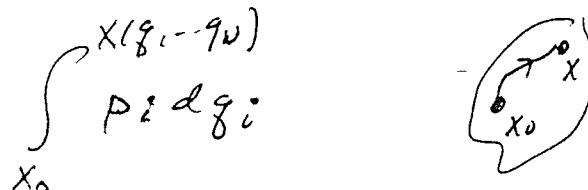
$$F_1 \rightarrow \left(\frac{\partial F_1}{\partial p_i}, -\frac{\partial F_1}{\partial q_i} \right) \quad F_2 \rightarrow \left(\frac{\partial F_2}{\partial p_i}, -\frac{\partial F_2}{\partial q_i} \right)$$

$$\begin{aligned} dp_i \wedge dq_i (\cdot, \cdot) &= -\frac{\partial F_1}{\partial q_i} \frac{\partial F_2}{\partial p_j} + \frac{\partial F_2}{\partial q_i} \frac{\partial F_1}{\partial p_j} \\ &= -[F_1, F_2] = 0 \end{aligned}$$

$\Rightarrow J_K = J_{K'} \dots$

on the torus M with fixed J_1, \dots, J_N , we choose Hamilton's characteristic function (generator of the canonical transformation) to be

$$W(q_1, \dots, q_N, J_1, \dots, J_N) = \int_{x_0}^{x(q_1, \dots, q_N)} p_i dq_i$$



This has the desired property

$$\frac{\partial \bar{W}}{\partial q_i} = p_i$$

and is used to define angle variables via

$$\frac{\partial \bar{W}}{\partial J_k} = w_k$$

(Note that, appealing to Stokes theorem again, we see that $\bar{W}(q, J)$ is independent of the path from x_0 to x on M .)

Finally, we observe that constructing action-angle variables involves only the "algebraic" step of solving for the (q, p) 's that solve $F_1, \dots, F_N = C_N$, and the evaluation of one-dimensional integrals $\int p_i dq_i$ on M — the problem has been shown to be "integrable by quadrature"

Canonical Perturbation Theory

Now we want to ask -- is integrability generic? Suppose H_0 is an integrable Hamiltonian. After a canonical transformation, it is $H_0(J) - \epsilon$ function of N action variables

Now we perturb H_0 by an infinitesimal amount

$$H(J, \omega) = H_0(J) + \epsilon H_1(J, \omega)$$

Perturbation is a function \nearrow
of J 's and ω 's

The trajectories governed by H_0 lie on invariant N -tori. For the nearly integrable motion governed by H , an orbit stays close to an invariant N -torus of H_0 for a long time. But does it stay close for an infinitely long time? It may eventually wander away, over a time that gets longer as ϵ gets smaller.

Our strategy for addressing the question is one that has been used for centuries in celestial mechanics. (It is called "time-independent perturbation theory" — see Goldstein, chapter 12.) We attempt to find, in a power series in ϵ , the distorted invariant tori of H that are close to the invariant tori of H_0 . In other words, we try to construct, perturbatively in ϵ , the canonical

Under this canonical transformation,

$$(w, J) \rightarrow (w', J')$$

The Hamiltonian becomes a function of the new action variables

$$H(J, w) = H'(J')$$

Let $G(w, J')$ be the generating function for this transformation

$$\frac{\partial G}{\partial w_i} = J_i \quad \frac{\partial G}{\partial J'_i} = w'_i$$

In zeroth order in ϵ , the transformation is the identity transformation. Expanding to linear order,

$$G = w_i J'_i + \epsilon G_1(w, J')$$

$$\text{so that } J_i = J'_i + \epsilon \frac{\partial}{\partial w_i} G_1(w, J')$$

$$w'_i = w_i + \epsilon \frac{\partial}{\partial J'_i} G_1(w, J')$$

We demand that

$$\begin{aligned} H(J, w) &= H_0(J' + \epsilon \frac{\partial}{\partial w_i} G_1) + \epsilon H_1(J', w) \\ &= H'(J') \end{aligned}$$

This becomes, to linear order in ϵ

$$H_0(J') + \epsilon \left(\frac{\partial}{\partial w_i} G_1 \right) \frac{\partial H_0}{\partial J'_i} + \epsilon H_1 = H'(J')$$

$$\text{Recall that } \frac{\partial H_0}{\partial J'_i} = \dot{w}_i = v_i(J)$$

(unquantized) frequency of the angle variable

11.45

so we have

$$H_0(J') + \epsilon v_i \frac{\partial}{\partial \omega_i} G_1(\omega, J') + \epsilon H_1(J', \omega) \\ = H'(J')$$

Now recall that each ω_i is a periodic variable with period 1, and expand in a Fourier series

$$H_1(J', \omega) = \sum_{\vec{m}} H_{1\vec{m}}(J') e^{2\pi i \vec{m} \cdot \vec{\omega}}$$

$$G_1(\omega, J') = \sum_{\vec{m} \neq 0} G_{1\vec{m}}(J') e^{2\pi i \vec{m} \cdot \vec{\omega}}$$

Here $\vec{m} = (m_1, m_2, \dots, m_N)$ represents N integers.
 (Note that the constant ($\vec{m} = 0$) term in G_1 does not contribute to $\frac{\partial}{\partial \omega_i} G_1$, so we are free to set it to zero.)

The condition satisfied by G_1 becomes

$$e^{2\pi i \vec{m} \cdot \vec{\omega}}$$

$$H_0 + \epsilon \sum_{\vec{m}} (2\pi i \vec{v} \cdot \vec{m} G_{1\vec{m}} + H_{1\vec{m}}) = H'$$

We equate Fourier coefficients on both sides:

$$\vec{m} = 0 \quad - \quad H' = H_0 + \epsilon H_{1\vec{0}}$$

(11.46)

(This determines the new Hamiltonian.)

$$\vec{m} \neq 0 \quad - \quad G_{\vec{m}} = \frac{-1}{2\pi i} \left(\frac{\vec{H}_{\vec{m}}}{\vec{v} \cdot \vec{m}} \right)$$

so we have constructed, to order ϵ , the canonical transformation

$$G = \vec{w} \cdot \vec{j}' - \frac{\epsilon}{2\pi i} \sum_{\vec{m} \neq 0} \frac{\vec{H}_{\vec{m}}}{\vec{v} \cdot \vec{m}} e^{2\pi i \vec{m} \cdot \vec{w}}$$

But there is a problem... Suppose that the N frequencies are commensurate — there is a linear combination of the ν_i 's, with integer coefficients, that vanishes. Then we have

$$\vec{v} \cdot \vec{m} = 0 \quad \text{for some } \vec{m} \neq 0$$

and one of the Fourier coefficients in the expansion of G_1 blows up. The perturbation expansion does not make sense...

Even if the frequencies are not commensurate, there will be choices of \vec{m} for which $\vec{v} \cdot \vec{m}$ is arbitrarily small. Therefore, no matter how small ϵ is, there will be terms in the first order correction to G that are very large. This suggests that higher order terms in ϵ can also be large — possibly ϵ^k are raised about the

(11.47)

convergence of the expansion in ϵ of each Fourier mode of G , and also about the convergence of the Fourier expansion itself.

This is the "problem of small divisors" that plagues perturbation theory. Note that it does not arise for systems with one degree of freedom — which are known to always be integrable.

What is going on is that systems with more than one degree of freedom have resonances, and resonance is the enemy of stability. For example, an integrable system with $N=2$ is like 2 uncoupled oscillators. But a generic perturbation couples these two oscillators together, however weakly. If the two oscillators have commensurate frequencies,

$$p\gamma_1 - q\gamma_2 = 0 \Rightarrow q\gamma_1^{-1} = p\gamma_2^{-1}$$

then oscillator 1 completes q cycles in the time that oscillator 2 completes p cycles. When the perturbation is turned on, this means that oscillator 1 kicks oscillator 2 at regular intervals (and vice versa). Though each kick is weak, the kicks add together in phase, and eventually accumulate to drive both 1 and 2 far from their original

11.48

In spite of the problem of small divisors, perturbation theory is sometimes useful. For a typical perturbation H_1 , the H_{lm} 's are small when the m 's are large, so we can truncate the Fourier expansion.

It may be that no small divisors occur for the values of m that are included. In practice then, the canonical transformation can be used to accurately predict orbits for a long time.

Example: For the sun-Jupiter-earth system, the unperturbed frequencies are in the ratio 11.86 to one. Many Fourier modes can be included before the large terms (small divisors) are encountered. Perturbation theory gives a good description of how Jupiter perturbs the motion of the earth around the sun, over many earth orbits. But it doesn't tell us how the perturbation influences the earth over an arbitrarily large no. of orbits.

What KAM managed to do was to find a much more powerful way of doing perturbation theory. crudely speaking, their idea is to use an iterative procedure, where the torus generated in step $n-1$ of the procedure is used as the starting point for the n th approximation (instead of finding the order ϵ^n term in the expansion about the next torus). This is roughly

analogous to Newton's method for finding the zero of a function, and has much better convergence properties than an expansion in power series about the unperturbed solution.

The main results are:

- For ϵ sufficiently small, the improved perturbation theory really does converge, "almost always" — so most trajectories remain confined to invariant N -tori for all time
- But all of the resonant tori with $\vec{m} \cdot \vec{\nu} = 0$ are destroyed! Trajectories that start out very near the resonant tori of the unperturbed system do not remain confined to N -tori — when the perturbation is turned on.

Together, these two results seem paradoxical. The resonant tori are dense in the phase space. If they are all destroyed, how can there be any invariant tori left? For, e.g., $N=2$, we have ratio...

$$\frac{\nu_1(J)}{\nu_2(J)}$$

— which varies smoothly as a function of the J_1, J_2 that label the tori

Resonant tori ($\nu_1/\nu_2 = \text{rational}$) are destroyed, $\nu_1 + \nu_2 = \text{integer}$

The resolution of the paradox is that the rational numbers, although dense, are countable. Hence there are sets that contain a neighbourhood of each rational number, yet have arbitrarily small measure.

What KAM really showed, for $N=2$, is that the tori survive if

$$\left| \frac{v_1}{v_2} - \frac{m}{n} \right| > \frac{K(\epsilon)}{n^{2.5}}, \quad \text{for all integers } m \text{ and } n,$$

where $K(\epsilon)$ is a function of ϵ (independent of m and n) that tends to 0 as $\epsilon \rightarrow 0$. (This function was not explicitly computed by KAM.) Analogous results were derived for all values of N .

The tori that are not known to survive are those with

$$\left| \frac{v_1}{v_2} - \frac{m}{n} \right| < \frac{K(\epsilon)}{n^{2.5}} \quad \text{for some } m \text{ and } n$$

Suppose $0 \leq v_1 \leq v_2$ so that $0 \leq \frac{v_1}{v_2} \leq 1$. How much of the total length of the unit interval is excluded by this condition?

We have length

$$L \leq \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{K}{n^{2.5}} = K \left(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \right) \sim K$$

(The sum over estimates L , since we

\uparrow
(Number of order 1)

So for $K \ll 1$, only a small "fraction" of order K of all tori are destroyed by the perturbation.

The KAM theorem does not explicitly say what happens to the orbits in the "gaps" between the surviving invariant tori. It does indicate that for ϵ fixed and small, these gaps are found near the resonant tori of the unperturbed problem with $\frac{\nu_1}{\nu_2} = \frac{m}{n}$

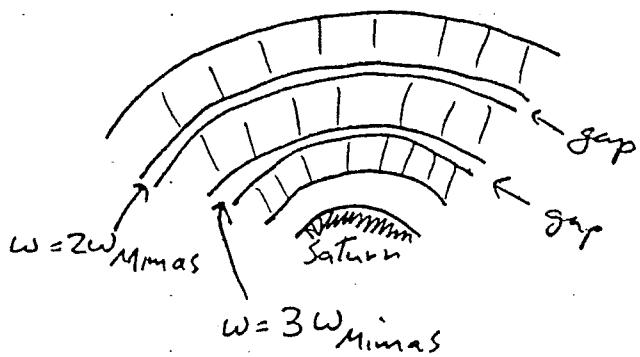
and are most prominent for small values of n .

Thus, the structure of the orbits for a slightly perturbed integrable system is remarkably intricate. The region of phase space in which tori are destroyed has a small volume, but is densely distributed in phase space! No wonder perturbation theory is hard!

In practice, the high order gaps ($n \gg 1$) may be very hard to observe, since they are very narrow. They are likely to be hidden by noise (random uncontrollable perturbations) or destroyed by dissipative effects.

A nice example of KAM gaps is provided by

← Mimas



The rings are composed of rocks moving in circular orbits around saturn. There are conspicuous gaps in the rings, clearly ring particle

The gaps occur where the orbits are in resonance with the orbits of various satellites of saturn. For example, two of the most conspicuous gaps occur where the orbits have frequency twice or three times the frequency of the orbit of the moon Mimas.

(You may be aware that there are many examples in the solar system of frequencies that are locked into resonance, contrary to KAM expectations. For example, the period of Mimas is $\frac{1}{2}$ the period of the satellite Tethys. The most famous example is the coincidence of the revolution and rotation periods of earth's moon. These coincidences are consequences of dissipative effects ("tidal friction") and so cannot be explained by Hamiltonian evolution.)

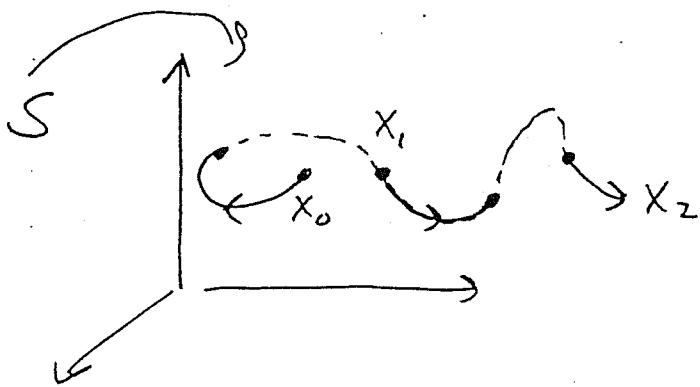
The Fate of Resonant Tori

Now we want to try to understand what happens, for a slightly perturbed integrable system, in the gaps between the invariant tori (at the gaps between the invariant tori).

We study this using the Poincaré surface of section" technique described on page 11.8.

For ease of visualization, we specialize to the case $N=2$ (two degrees of freedom).

The condition $H=E=\text{constant}$ determines a 3-d submanifold of the 4-d phase space.



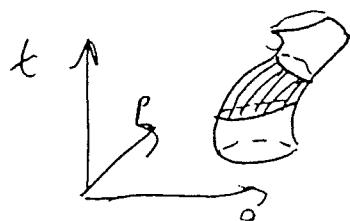
Now choose a 2-d surface S embedded in the $H=E$ manifold. The orbits of the Hamiltonian system intersect S at isolated points

Successive points where the orbit pierces through S in the same sense are successive iterates of the map. (The time interval associated with map agrees with the point x in S .) This map

$$M: S \rightarrow S$$

is the Poincaré map.

The Poincaré map has a very important property — it is an area-preserving two-dimensional map. This follows from the Poincaré-Cartan theorem discussed on pp 7A.14 ff.



Recall that for a "Tube" of trajectories in extended phase space, we have that ...

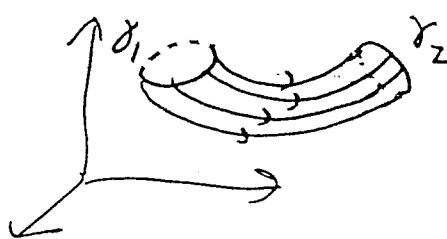
$$\oint_{\gamma_1} (p_i dq_i - H dt) = \oint_{\gamma_2} p_i dq_i - H dt$$

for any two closed paths γ_1 and γ_2 in extended phase space that enclose the tube. If this tube is contained in the surface with $H = E = \text{constant}$, we have

$$\oint_{\gamma} H dt = E \oint_{\gamma} dt = 0,$$

so Poincaré-Cartan becomes

$$\oint_{\gamma_1} p_i dq_i = \oint_{\gamma_2} p_i dq_i$$



Now choose γ_1 and γ_2 to be intersections of the tube of trajectories with the surface S .

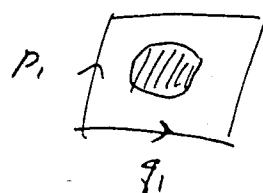
Then above says that phase space area

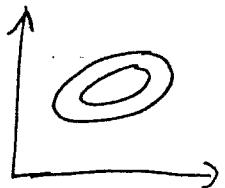
$$\oint_{\gamma} p_i dq_i = \int_S dP_1 \wedge dq_1$$

is preserved by the map

E.g. if we choose S to be the surface $q_2 = 0$, and parametrize S by q_1, p_1 , this area element is

$$\oint p_i dq_i$$



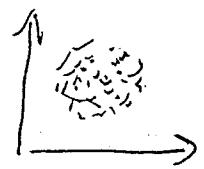


By studying the Poincaré map, we can tell whether the system is integrable. Invariant tori will generically

intersect the surface S (with a particular sense) on a closed curve. So orbits of the Poincaré map are typically dense on a closed curve, for integrable systems.

If the system is ergodic, the orbit will tend to fill S . More generally,

if orbit of the Hamiltonian system is not confined to an invariant torus, the orbit of the Poincaré map will tend to fill a two-dimensional region.



Example: Hénon - Heiles Potential

As an example of a nonintegrable system, consider a particle in two-dimensions moving in a potential

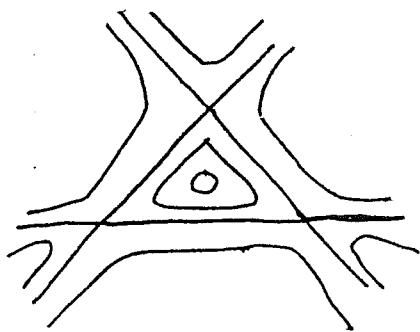
$$H = \frac{1}{2}(P_x^2 + P_y^2) + V$$

$$V = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

In polar coordinates:

$$\begin{aligned} x^2y - \frac{1}{3}y^3 &= r^3(\cos^2\theta \sin\theta - \frac{1}{3}\sin^3\theta) \\ &= \frac{1}{3}r^3((\cos^2\theta - \sin^2\theta)\sin\theta \\ &\quad + 2\sin\theta \cos\theta \cos\theta) \\ &= \frac{1}{3}r^3 \sin 3\theta \end{aligned}$$

$$\text{So } V = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin 3\theta$$



Equipotentials look as shown
There is a local minimum
at $r=0, V=0$

$$\text{on line } x=0 \quad V = \frac{1}{2}y^2 - \frac{1}{3}y^3$$

- local maximum occurs $y=1, V=\frac{1}{6}$

$$\text{Also } V = \frac{1}{2}(x^2+y^2) + x^2y - \frac{1}{3}y^3$$

is independent of x on line $y=-\frac{1}{2}$,

$$V = \frac{1}{6}$$

~~$\frac{1}{6}x^2 + \frac{1}{3}y^3$~~

From 3-fold rotational symmetry, we can infer
shape of equipotentials - the energy surface
is bounded only for $E < \frac{1}{6}$

For $0 < E < \frac{1}{6}$ - the potential is very
nearly harmonic, and the cubic term
can be regarded as a small perturbation

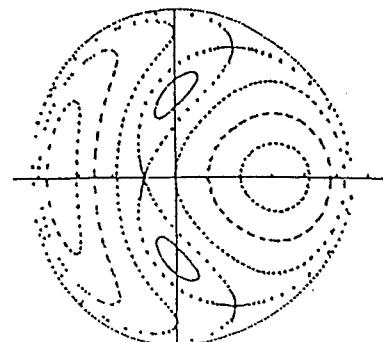
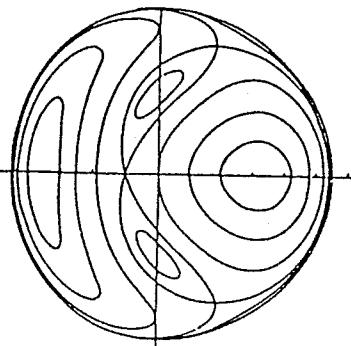
Numerical calculations of the Poincaré
map for this system are represented
in the picture on the next page.

These are orbits of the map, shown in
the x, p_x plane. On the left are the
perturbed tori found in perturbation theory,
carried out to eighth order.

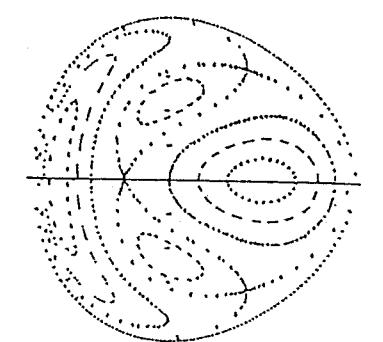
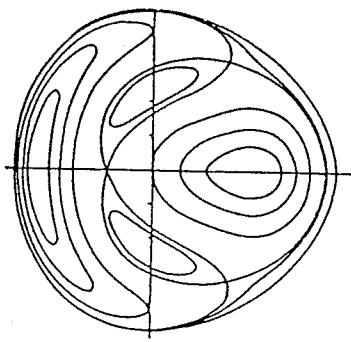
For $E \approx \frac{1}{9}$. The orbits seem to lie on
invariant curves, those predicted by the
perturbation theory. But for $E = \frac{1}{8}$ and $E = \frac{1}{6}$,
the destruction of the tori is apparent.
In fact, all the points for $E = \frac{1}{8}$ and $E = \frac{1}{6}$ were generated
in + - + - + - ...

(11.57)

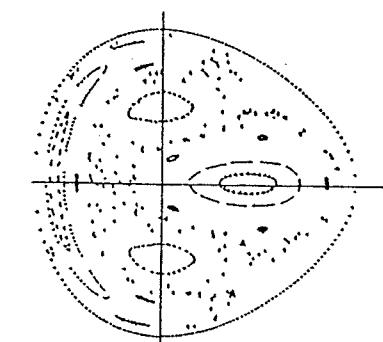
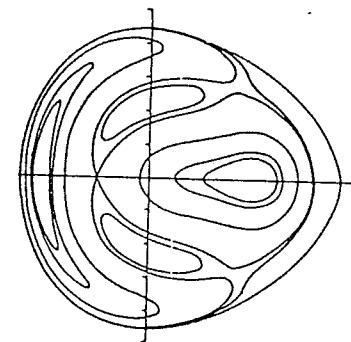
$$E = \frac{1}{24}$$



$$E = \frac{1}{12}$$



$$E = \frac{1}{8}$$



$$E = \frac{1}{6}$$

