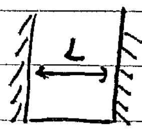
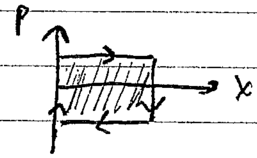
Physics 106a
Problem Set 7 Solution

(1) a) 

Inside box, the particle's momentum is $p = \pm\sqrt{2mE}$



The action is $J = \oint p\,dq = 2L\sqrt{2mE}$

$$\frac{dJ}{dE} = 2L\sqrt{2m}\left(\tfrac{1}{2}E^{-\frac{1}{2}}\right) = \frac{2mL}{|p|} = \frac{2L}{v}$$

where $v = |p|/m$ is the speed of the particle, so $T = \frac{2L}{v} = \frac{dJ}{dE}$

b) The impulse $\Delta p = 2|p| = 2\sqrt{2mE}$ occurs once per period T

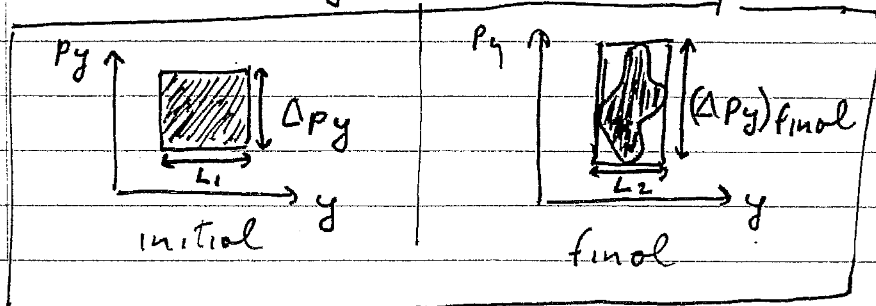
$$\Rightarrow \langle F \rangle = \frac{\Delta p}{T} = 2|p|\left(\frac{v}{2L}\right) = \frac{p^2}{mL} = \frac{2E}{L}$$

Express in terms of L and the adiabatic invariant J:

$J^2 = 8mL^2E \Rightarrow$ $\boxed{\langle F \rangle = \frac{J^2}{4mL^3}}$ Thus $\langle F \rangle \propto \frac{1}{L^3}$

(2)

We treat the beam as an ensemble of one-particle systems. Motion in $\hat{x}$ and $\hat{y}$ directions separate, so we may consider only the y motion.



The initial phase space distribution fills (perhaps not uniformly) a rectangle with dimensions $L_1 \times \Delta p_y$. By Liouville's theorem, this rectangle evolves to a new shape with the same volume. The final shape is contained in a rectangle with dimensions $L_2 \times (\Delta p_y)_{final}$. Therefore

$$L_2 \cdot (\Delta p_y)_{final} \geq L_1 (\Delta p_y) \Rightarrow \boxed{(\Delta p_y)_{final} \geq \frac{L_1}{L_2} \Delta p_y}$$

3)

$$Q = q^{\alpha} \cos \beta p$$
$$P = q^{\alpha} \sin \beta p$$

Compute Poisson Bracket:

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \alpha \beta q^{2\alpha - 1}$$

Thus, demanding $[Q, P] = 1$, we have $\boxed{\alpha = \tfrac{1}{2}, \ \beta = 2.}$

We have, now,

$$Q = q^{\frac{1}{2}} \cos 2p \qquad \Rightarrow \qquad q = P^2 + Q^2$$
$$P = q^{\frac{1}{2}} \sin 2p \qquad\qquad \tan 2p = P/Q$$

Since..

$$P = Q \tan 2p$$
$$q = Q^2 (1 + \tan^2 2p) = Q^2 \sec^2 2p$$

it is convenient to construct

$F_3(p, Q)$, with

$$\frac{\partial F_3}{\partial p} = -q = -Q^2 \sec^2 2p,$$
$$\frac{\partial F_3}{\partial Q} = -P = -Q \tanh 2p.$$

Integrating, we find $\boxed{F_3(p, Q) = -\tfrac{1}{2} Q^2 \tan(2p)}$

4 $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$

$\Longrightarrow \quad p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad p_\phi = mr^2\sin^2\theta\,\dot{\phi}$

so $H = \frac{1}{2m}\left(p_r^2 + p_\theta^2/r^2 + p_\phi^2/r^2\sin^2\theta\right) + V$

Now let $S = W - Et$, and H-J eqn. becomes

$E = H(q_i, \frac{\partial W}{\partial q_i})$, or

$$\boxed{E = \left[\frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + V_r(r)\right] + \frac{1}{r^2}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial \theta}\right)^2 + V_\theta(\theta)\right] + \frac{1}{r^2\sin^2\theta}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial \phi}\right)^2 + V_\phi(\phi)\right]}$$

If

$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$,

we have a solution provided...

$$\frac{1}{2m}\left(\frac{dW_\phi}{d\phi}\right)^2 + V_\phi(\phi) = P_\phi = \text{constant}$$

$$\frac{1}{2m}\left(\frac{dW_\theta}{d\theta}\right)^2 + V_\theta(\theta) + P_\phi/\sin^2\theta = P_\theta = \text{constant}$$

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + V_r(r) + P_\theta/r^2 = E = \text{constant}$$

Integrating :

$$W_\phi = (2m)^{\frac{1}{2}} \int d\phi \, (P_\phi - V_\phi)^{\frac{1}{2}}$$

$$W_\Theta = (2m)^{\frac{1}{2}} \int d\Theta \, (P_\Theta - V_\Theta - P_\phi / \sin^2\Theta)^{\frac{1}{2}}$$

$$W_r = (2m)^{\frac{1}{2}} \int dr \, (E - V_r - P_\Theta / r^2)^{\frac{1}{2}}$$

To find general solution:

$$\boxed{\text{constant} = Q_E = \frac{\partial S}{\partial E} = -t + (\tfrac{1}{2}m)^{\frac{1}{2}} \int dr \, (E - V_r - P_\Theta/r^2)^{-\frac{1}{2}}}$$

This (implicitly) determines $r(t)$.

$$\boxed{\begin{aligned} \text{constant} = Q_\Theta = \frac{\partial S}{\partial P_\Theta} = (\tfrac{1}{2}m)^{\frac{1}{2}} \Big\{ & - \int dr \, r^{-2} (E - V_r - P_\Theta/r^2)^{-\frac{1}{2}} \\ & + \int d\Theta \, (P_\Theta - V_\Theta - P_\phi/\sin^2\Theta)^{-\frac{1}{2}} \Big\} \end{aligned}}$$

This determines $\Theta(r)$.

$$\boxed{\begin{aligned} \text{constant} = Q_\phi = \frac{\partial S}{\partial P_\phi} = (\tfrac{1}{2}m)^{\frac{1}{2}} \Big\{ & - \int d\Theta \, \sin^{-2}\Theta \, (P_\Theta - V_\Theta - P_\phi/\sin^2\Theta)^{-\frac{1}{2}} \\ & + \int d\phi \, (P_\phi - V_\phi)^{-\frac{1}{2}} \Big\} \end{aligned}}$$

This determines $\phi(\Theta)$.

So we have found the general solution for r, Θ, ϕ in terms of t and the arbitrary constants

$E = P_r , P_\Theta , P_\phi , Q_E , Q_\Theta , Q_\phi$.