Ph 106a

Homework Assignment No. 6
Due: Thursday, 29 November 2007

1. Normal modes. For the mechanical system (ii) of problem 3 on Homework No. 5, find the normal mode frequencies $\omega_a$, the normal mode vectors $\vec{X}^{(a)}$, and the normal coordinates $Q_a$, such that

$$\eta_i = \sum_a X_i^{(a)} Q_a$$

and

$$L = \frac{1}{2} \sum_{i,j} (T_{ij} \eta_i \dot{\eta}_j - V_{ij} \eta_i \eta_j)$$
$$= \frac{1}{2} \sum_a (\dot{Q}_a^2 - \omega_a^2 Q_a^2) .$$

Verify that the normal mode vectors obey the orthogonality condition

$$\sum_{i,j} X_i^{(a)} T_{ij} X_j^{(b)} = \delta^{ab} .$$

Define your generalized coordinates $\eta_i$ as in the solution to last week’s problem 3, and use the matrices $T_{ij}$ and $V_{ij}$ found in that solution.

2. Masses and springs. Same as problem (1) above, but for the mechanical system (iii) of problem 4 on Homework No. 5.

3. Action as a function of the coordinates. For a free particle of mass $m$ in one dimension, calculate the action $S$ of a trajectory that solves the equation of motion, where the particle begins at position $x_1$ at time $t_1$ and ends at position $x_2$ at time $t_2$. Verify the relations derived in class:

$$\frac{\partial S}{\partial x_2} = p(t_2) , \quad \frac{\partial S}{\partial t_2} = -H(t_2) .$$
4. **Virial theorem.** Consider a gas of $N$ particles which interact gravitationally in three dimensions. Assume that the gas is in equilibrium so that

$$\left\langle \frac{d}{dt} \sum_i p_i q_i \right\rangle = 0,$$

where $\langle \cdot \rangle$ denotes the time average. Use the virial theorem in the form

$$\langle T \rangle = \frac{1}{2} \left\langle \sum_i q_i \frac{\partial H}{\partial q_i} \right\rangle$$

(6)

to find a relation between the time-averaged quantities $\langle T \rangle$ and $\langle V \rangle$, where the potential energy $V$ is defined so that it vanishes when each particle is infinitely far from all the others.