1. **Sleeping top.** In class we considered the “sleeping” top, the symmetrical top with one point fixed and initial conditions $\theta = \dot{\theta} = 0$. We saw that if the angular velocity $\omega_s$ of the top exceeds the critical value $\omega_{\text{crit}} = (2/I_3)(Mg\ell I)^{1/2}$, then the vertical position of the top is stable.

   a) If a sleeping top with $\omega_s > \omega_{\text{crit}}$ receives a small horizontal impulse, then the symmetry axis of the top will execute small oscillations about the vertical. That is, the inclination $\theta$ from the vertical will vary as
   
   $$\theta \approx \theta_0 \cos(\omega t).$$
   
   where $\theta_0$ is a small constant that depends on the size of the impulse. Express the circular frequency $\omega$ of these oscillations in terms of $\omega_s$, $\omega_{\text{crit}}$, $I_3$, and $I$.

   b) When the (stable) vertical sleeping top receives a small horizontal impulse, it returns to the vertical after time $t_0 = \pi/\omega$. By what amount $\Delta \phi$ does the Euler angle $\phi$ (the azimuthal position of the top’s symmetry axis) advance in the time $t_0$? What is $\Delta \phi$ in the limit of a fast top with $\omega_s \gg \omega_{\text{crit}}$?

2. **Sliding top.** Suppose the axis of a symmetrical top ends at a contact point that slides without friction on a horizontal table. The top has mass $M$, $\ell$ is the distance from the contact point to the top’s center of mass, $I_3$ is the top’s moment of inertia for rotations about it’s symmetry axis, and $I$ is the moment of inertia for rotations about an axis perpendicular to the symmetry axis that passes through the top’s center of mass.

   a) Using as generalized coordinates the $x$ and $y$ coordinates of the top’s center of mass, and the three Euler angles $\theta$, $\phi$, $\psi$ describing rotations of the top about it’s center of mass, write down the Lagrangian for this system.

   b). Using conservation laws, find a function $g(u)$ such that
   
   $$u^2 = g(u),$$
   
   where $u = \cos \theta$. 
3. Small oscillations. For the mechanical systems (i) and (ii) illustrated on page 3:

a) Choose suitable generalized coordinates and find the Lagrangian.
b) Find the equilibrium positions.
c) Expand to quadratic order to find matrices $T_{ij}$ and $V_{ij}$ such that the Lagrangian can be expressed as

$$L = \frac{1}{2} \sum_{i,j} (T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j) + \cdots ,$$

where $\eta_i$ is the deviation of the $i$th generalized coordinate from its equilibrium position.

(For each spring, the natural unstretched length is $L_0$.)

4. More small oscillations. Same as problem 3, for mechanical system (iii) illustrated on page 3.
Illustrations for Problems 3 and 4:

(i) (Motion confined to the plane. Pivots of both pendula are at the same height.)

(ii) (Motion confined to one dimension.)

(iii) (Motion confined to the plane.)