Physics 236A, Fall 2002
Problem Set 7

Date assigned: November 15 **Due on November 22 by 2:30pm**

Reading Assignments:

Read Sections 4.3 and 4.4 of Wald.

Homework Problems:

[1] In the lecture, we determined the energy-momentum tensor for the free scalar field as

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial^\rho \phi \partial_\rho \phi + m^2 \phi^2).$$

In n-dimensional spacetime, the trace of this energy-momentum tensor is

$$T^\mu_{\ \mu} = \frac{1}{2} \left[(2-n) \partial^\rho \phi \partial_\rho \phi - n m^2 \phi^2 \right].$$

This does not vanish even if the field is massless $m = 0$, unless $n = 2$ (two-dimensional spacetime). In particular, the energy-momentum tensor of the massless scalar field in four-dimensional spacetime is not traceless, and the model does not have a local scale invariance.

One can *improve* the situation by adding a coupling of ϕ to the scalar curvature R as

$$S = -\frac{1}{2} \int d^n x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \alpha R \phi^2),$$

where α is a constant. Show that, for a certain value of α, the action is invariant under the local scale transformation,

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu},$$

if we transform the scalar field simultaneously as

$$\phi(x) \to \Omega^{(2-n)/2}(x) \phi(x).$$

Determine the value of α.

Compute the energy-momentum tensor for this value of α and verify that it is conserved and that it is traceless.

One may use the formula (D.8) of Appendix D of Wald.

[2] The units in which $G = 1$ and $c = 1$ are called geometrized units. For example, $G/c^2 = 7.425 \times 10^{-28}$m/kg and it is set equal to 1 in geometrized units. There are some advantages of using these units, besides simplification of various formulae. For example, the mass of Earth $M_\oplus$ is measured by examining satellite orbits and using Kepler's laws. This actually measures $GM_\oplus$ rather than $M_\oplus$. This quantity is known to an accuracy of five parts in 10^8. On the other hand, G/c^2 is measured in laboratory experiments and is uncertain by five parts in 10^4. So, the SI value of $M_\oplus$ is also known only at this accuracy.

$$c = 2.998 \times 10^8 \text{m/s}, \quad G = 6.673 \times 10^{-11} \text{m}^3/\text{kgs}^2.$$

Express the following quantities in geometrized units.

(1) a density of typical neutron stars, $\rho = 10^{17}\text{kg/m}^3$.

(2) a pressure of typical neutron stars, $p = 10^{33}\text{kg/ms}^2$.

(3) the acceleration of gravity on Earth's surface, $g = 9.80\text{m/s}^2$.

(4) the luminosity of a supernova, $L = 10^{41}\text{J/s}$.

(5) the Newtonian potential of the Sun at the Sun's surface, radius 6.960×10^8m.

(6) the Newtonian potential of the Sun at the radius of Earth's orbit, $1\text{AU} = 1.496 \times 10^{11}$m.

(The solar mass is 1.989×10^{30} kg.)

[3] This is to complete the analysis of the gravitational wave polarization in the lecture. Consider the plane wave of the form,

$$\bar{h}_{\mu\nu}(x) = C_{\mu\nu} e^{ikx},$$

and choose coordinates so that $k = (\omega, 0, 0, \omega)$. Impose the gravitational Lorentz gauge $k^\mu C_{\mu\nu} = 0$. Show that, by using the residual gauge invariance, we can set $C_{0\mu} = C_{3\mu} = 0$ and $C_{11} + C_{22} = 0$, leaving $C_{11}$and C_{12} as only two independent components.