Physics 231, Spring 1999
Problem Set 6

Date assigned: November 8 **Due on November 15 by 2:30**

Announcement: There will be no class meeting on November 26 (Tuesday).

Reading Assignments:

Study Sections 3.4 and 4.3 of Wald. We did not discuss the tetrad method since we will not use it in this course. It is discussed in Section 3.4b and I recommend and encourage you to study this section also.

Homework Problems:

[1] Riemann normal coordinates $(\xi^1, ..., \xi^n)$ in a neighborhood of a point p in M is defined as follows. Pick a tangent vector $\xi \in T_pM$ and find a geodesic $x(s)$ with the initial condition

$$x(0) = p, \quad \frac{dx^\mu}{ds}_{|s=0} = \xi^\mu.$$

We then define the exponential map, *Exp*, as

$$\xi \in T_pM \quad \rightarrow \quad Exp(\xi) = x(1) \in M.$$

If the manifold M is geodesically complete, *Exp* is defined for any tangent vector ξ. Otherwise, we may have to limit to a subspace V of T_pM, on which $Exp(\xi)$ is non-singular. Since $T_pM \sim R^n$, its subspace V is an open subset of R^n. We can then use $\xi \in V$ as coordinates in the neighborhood of p, which is the image of V.

(a) Show that, in the normal coordinates, the Christoffel connection $\Gamma^\rho_{\mu\nu}$ vanishes at p, although its derivatives may not vanish.

(b) Prove the Bianchi identity at p using the normal coordinates. Since the identify is independent of coordinates, this is sufficient as a proof of the identity.

[2] We would like to show that, if the Riemann curvature $R_{\mu\nu\rho}{}^\sigma$ vanishes in some neighborhood U of a point p, we can always find coordinates such that the metric tensor $g_{\mu\nu}$ takes the form $\eta_{\mu\nu} = diag(-1, +1, ..., +1)$ (or $\delta_{\mu\nu} = diag(+1, +1, ..., +1)$ in the case of the Euclidean signature metric), not only at the point p (which is

always possible), but everywhere in the neighborhood U. Namely $R_{\mu\nu\rho}^{\ \ \ \ \sigma} = 0$ means the space is flat. Prove this statement in the following steps.

(a) Show that we can find n linearly independent cotangent vector fields $\omega_\mu^{(i)}(x)$ $(i = 1, ..., n)$ which are *covariantly constant*, *i.e.* $\nabla_\mu \omega_\nu^{(i)} = 0$.

(b) Show that the result of (a) means they satisfy $\partial_{[\mu,}\omega_{\nu]}^{(i)} = 0$ and that we can find functions $f^{(i)}(x)$ such that $\omega_\mu^{(i)} = \partial_\mu f^{(i)}(x)$.

(c) Show that, if we use $y^i = f^{(i)}(x)$ $(i = 1, ..., n)$ as coordinates, the metric is expresses as a constant tensor in the neighborhood U. Therefore, by an appropriate linear transformation in y^i, the metric is expressed in the flat form.

[3] Under a coordinate transformation, $x^\mu \to \tilde{x}^\mu$, the metric tensor transforms as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x).$$

Show that, for an infinitesimal transformation, $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, this takes the form,

$$\delta g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu).$$

In particular, if the transformation preserves the metric if and only if

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 0.$$

A solution $\epsilon(x)$ to this equation is called a Killing vector field.