Physics 231, Spring 1999
Problem Set 5

Date assigned: November 1 **Due on November 8 by 2:30**

Announcement: There will be no class meeting on November 26 (Tuesday).

Reading Assignments:
Study Sections 3.1, 3.2, and 3.3 of Wald.

Homework Problems:

[1] A two-dimensional sphere S^2 of radius r can be embedded in the three- dimensional Euclidean space R^3 by the equation,

$$x^2 + y^2 + z^2 = r^2.$$

We can use (θ, ϕ) defined by

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta,$$

as coordinates on the sphere, except at the north and south poles $\theta = 0, \pi$, where the value of ϕ is ambiguous.
(a) An infinitesimal rotation of R^3 around its origin induce tangent vector fields on S^2. Show that there are three linearly independent vector fields of this type and compute their commutators, [,].
(b) The standard Euclidean metric $dx^2 + dy^2 + dz^2$ induces the metric,

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Compute the Riemann curvature tensor, the Ricci curvature, and the scalar curvature.
Show that the surface integral of the scalar curvature R,

$$\int_{S^2} \sqrt{g} d\theta d\phi R,$$

over the 2-sphere is independent of the radius r.

(c) The geodesics on the two sphere are clearly the great circles. By looking at the relative acceleration of neighboring great circles and comparing with the geodesic deviation equation (see section 3.3 of Wald), arrive at the same answer as (b).

[2] Using the coordinate transformation property of the connection $\Gamma_{\mu}{}^{\rho}{}_{\nu}$, verify that the covariant derivative of a tensor field $T^{\mu}_{\ \nu}$ defined by

$$\nabla_{\mu} T^{\nu}{}_{\rho} = \partial_{\mu} T^{\nu}{}_{\rho} + \Gamma_{\mu}{}^{\nu}{}_{\sigma} T^{\sigma}{}_{\rho} - \Gamma_{\mu}{}^{\sigma}{}_{\rho} T^{\nu}{}_{\sigma}$$

transforms as a tensor field.

[3] Do Problem 4 (a) of Chapter 3 of Wald. Verify that the general expression for the curvature in two dimensions is consistent with the result of the problem [1] (b) in the above.

[4] Two very small vector fields U^{μ} and V^{μ} are used to construct a very small parallelogram with sides: $U^{\mu}, V^{\mu}, -U^{\mu}, -V^{\mu}$. Parallel transport a vector A^{μ} around this path and show that upon return it has changed by $\delta A^{\mu} = R^{\sigma}_{\mu\nu\rho} U^{\mu} V^{\nu} A^{\rho}$. Show that its length has not changed and hence it was simply being rotated.