Physics 236A, Fall 2002
Problem Set 3

Date assigned: October 18 **Due on October 25 by noon**

Reading Assignments:
Study Sections 2.1 of Wald.

Further Reading:
To learn more about manifolds and Riemannian geometry, I recommend Chapters 5, 6, and 7 of "Geometry, Topology and Physics" by M. Nakahara (Adam Hilger). Another good reading would be "Geometrical Methods of Mathematical Physics" by B. Schultz (Cambridge U. Press).
For advanced reading on special relativity, look at "The Classical Theory of Fields" by L.D. Landau and E.M. Lifshitz.

Homework Problems:

[1] In section 4.2 of Wald, it is claimed that the conservation, of the energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{c}\big((\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu}\big)$$

of a perfect fluid implies, in the non-relativistic limit $\epsilon \gg p$, the conservation of mass and Euler's equation (equations (4.2.14) and (4.2.15) in the section). Verify this statement explicitly by taking the non-relativistic limit of $\partial_\mu T^{\mu\nu} = 0$. (Note that Wald uses ρ for ϵ and set the speed of light $c = 1$.)

[2] (I am setting $c = 1$ in this problem.)

(2.1) Show that the Klein-Gordon equation for a massive real scalar field,

$$\partial^\mu \partial_\mu \phi - m^2 \phi^2 = 0,$$

is obtained by applying the variational principle to the action

$$S = \frac{1}{2}\int d^4x(-\partial_\mu\phi\partial^\mu\phi - m^2\phi^2).$$

(2.2) Find the energy-momentum tensor $T^{\mu\nu}$ of ϕ. Check the conservation of $T^{\mu\nu}$ explicitly using the Klein-Gordon equation. Find expressions for the total energy

$$E = \int d^3\vec{x} c T^{00},$$

and the total momentum momentum

$$P^i = \int d^3\vec{x} T^{0i}$$

of the scalar field ϕ.

[3] Consider a *complex* scalar field φ with an electric charge e. The total action is given by

$$S_{total} = S_{EM} + \int d^4x \big(- [(\partial_\mu + ieA_\mu)\bar{\varphi}][(\partial^\mu - ieA^\mu)\phi] - m^2\bar{\varphi}\varphi \big),$$

where S_{EM} is the action for the electromagnetic vector potential A_μ. Derive the equations of motion for φ and A_μ using the variational principle. Obtain the total energy-momentum tensors and show that it is conserved using the equations of motion.

[4] Use the conservation of the energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$, prove the following results for a bounded system, *i.e.* a system for which $T^{\mu\nu} = 0$ outside a bounded regions of space,

$$\frac{\partial^2}{\partial t^2} \int T^{00} x^i x^j d^3x = 2 \int T^{ij} d^3x, \qquad \text{(called virial theorem)},$$

$$\frac{\partial^2}{\partial t^2} \int T^{00} (x^i x^i)^2 d^3x = \int \left(4T^{ii} x^j x^j + 8T^{ij} x^i x^j\right) d^3x.$$

[5]

(4.1) Out of the three physical constant– h (Planck constant), G (gravitational constant), and c (speed of light) — form natural units of mass, length, and time.

(4.2) Estimate the Bohr radius of a "gravitational atom", *e.g.* two newtrons bound by their gravitational attraction (ignoring nuclear forces) in their lowest energy state.

[6] Do Problem 1 of Wald's Chapter 2.