Physics 236A, Fall 2002

Problem Set 1

Date assigned: October 11 **Due on October 18 by noon**

Bring the homework to 471 Lauritsen, attn: Lee/Park.

Reading Assignments:

Read Sections 4.1 and 4.2 of Wald.

Homework Problems:

[1] Show that the first set of the Maxwell equation

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$$

is invariant under the general coordinate transformation,

$$x^\mu \to \tilde{x}^\mu = f^\mu(x), \qquad F_{\mu\nu}(x) \to \tilde{F}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}F_{\rho\sigma}(x),$$

for an arbitrary $f^\mu(x)$ with non-zero Jacobian.

[2] It has been known for a long time that the vacuum Maxwell equation,

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = 0,$$

is invariant under the exchange of the electric and the magnetic fields

$$(E, H) \to (H, -E) \qquad (1).$$

This is known as the electromagnetic duality. If we include electric charges, the Maxwell equation is invariant only if we introduce the concept of the magnetic monopole charge and exchange the electric and the magnetic monopole charges simultaneously under the duality transformation (This observation motivated Paul A. M. Dirac to study properties of the magnetic monopole and discover the famous quantization condition $eg = \frac{n}{4\pi}hc$ between the electric charge e and the magnetic charge g, where $n = 0, \pm 1, \pm 2, \ldots$. See *Phys. Rev.* 74 (1948) 817.).

Show that the duality transformation (1) can be written in the Lorentz covariant fashion as

$$F_{\mu\nu} \to \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}.$$

Verify that the vacuum Maxwell equation is indeed invariant under this transformation.

[3] Show that $\vec{E}^2 - \vec{H}^2$ and $\vec{E}\cdot\vec{H}$ are scalar and pseudo-scalar under the Lorentz transformation by expressing them in terms of $F_{\mu\nu}$.

[4] Show the following two facts which we used in the class but did not derive.

(4.1) For the Lagrangian density $\mathcal{L}(\phi, \partial\phi)$ of a scalar field ϕ, define the energy-momentum tensor as

$$T^{\mu}_{\ \nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi + \delta^{\mu}_{\nu}\mathcal{L}.$$

Show that the equation of motion implies the conservation $\partial_\mu T^{\mu}_{\ \nu} = 0$.

(4.2) Show that the energy-momentum tensor of the Maxwell field,

$$T^{\mu\nu}_{EM} = \frac{1}{4\pi c}(F^{\mu\rho}F^{\nu}_{\ \rho} - \frac{1}{4}\eta^{\mu\nu}F^2),$$

in the presence of the electric current j^μ, obeys

$$\partial_\mu T^{\mu\nu}_{EM} = -\frac{1}{c}j^\rho F^{\nu}_{\ \rho},$$

making use of the Maxwell equations.

(Extra credit) Give a physical interpretation of this equation. Interpreting T^{00} as energy density and T^{0i} as momentum density, explain how the presence of the electric current j^ρ takes up energy and/or momentum.

[5]

(5.1) Show that, in Minkowski space, the general solution of the equation

$$\partial_{(\mu,}\xi_{\nu)} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$$

is given by $\xi_\mu = M_{\mu\nu}x^\nu + P_\mu$ where $M_{\mu\nu}$ and P_μ are constant and $M_{\mu\nu} = -M_{\nu\mu}$.

(5.2) How about the general solution to

$$\partial_{(\mu,}\xi_{\nu)} = c(x)\eta_{\mu\nu} \qquad (2)$$

in four-dimensional Minkowski space, with an undetermined function $c(x)$?

(*Hint*: Taking a trace over both sides of (2) gives $c(x)$ expressed in terms of ξ.)

(5.3) If we consider the same equation (2) in two spacetime dimensions, something special happens. What is that?