Physics 129c, Spring 2002

Final Examination

Due on June 3 (Monday) at noon to 422 Downs, attn: Takuya Okuda

This is a closed-book three-hour take-home examination. You are allowed to use a calculator and a table of functions. Please try to answer all questions, and show your work to maximize the possibility of partial credit. Read all the problems before you start solving them. The maximum points of each question are indicated.

[1] (a) Show that there are only two groups of order 4 and that both groups are Abelian. (*10 points*)
(b) Find all the irreducible representations for both groups. (*10 points*)

[2] The Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider

$$X = \sum_{i=1}^{3} x_i \sigma_i,$$

for some variables (x_1, x_2, x_3).

(a) Show $\det X = -\sum_i x_i^2$. (*6 points*)
(b) Show X is traceless and hermitian. Show that any traceless hermitian 2×2 matrix can be expressed in the form $\sum_i x_i \sigma_i$. (*6 points*)
(c) Show that, for any $U \in SU(2)$, the combination UXU^{-1} is tracelss and hermitian. (*6 points*)
(d) Let (x_1', x_2', x_3') be defined by

$$\sum_i x_i' \sigma_i = U \sum_i x_i \sigma_i U^{-1}.$$

Show that this defines a homomorphism map from $SU(2)$ to $SO(3)$ via $x_i \to x_i'$. (*6 points*)
(e) Find the kernel of this homomorphism. (*6 points*)

[3] Do the following matrices form a group?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$ and the group multiplication is defined as the matrix multiplication. If they don't, find other 2×2 matrices needed to complete a group with smallest order possible. Divide the elements of the group into conjugacy classes. (*20 points*)

[4] Consider three pairs of creation and annihilation operators

$$[a_i, a_j^\dagger] = \delta_{ij}, \tag{1}$$

$$[a_i, a_j] = 0, [a_i^\dagger, a_j^\dagger] = 0, \tag{2}$$

$$a_i|0\rangle = 0, i, j = 1, 2, 3, \tag{3}$$

and unitary operators $U(\sigma)(\sigma \in S_3)$ satisfying

$$U(\sigma) a_i^\dagger U(\sigma)^{-1} = a_{\sigma(i)}^\dagger. \tag{4}$$

Let V be a vector space spanned by the states

$$|ij\rangle \equiv a_i^\dagger a_j^\dagger |0\rangle.$$

(a) What is the dimension of V? Show that $U(\sigma)$ give a representation of S_3 on V. (*10 points*)
(b) Compute the characters of this representation. (*10 points*)
(c) Calculate the number of times each irreducible representation occurs in the representation on V. (*10 points*)

Hint

The character table of S_3 was worked out in class and is given by

	C_1	C_2	C_3
A_1	1	1	1
A_2	1	-1	1
E	2	0	-1

Here $C_1 = \{e\}, C_2 = \{(12), (23), (31)\}$ and $C_3 = \{(123), (132)\}$ are the conjugacy classes of S_3, and A_1 is the trivial representation, A_2 is another one-dimensional representation, and E is a two-dimensional irreducible representation.