

# Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models

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# Principal $\sigma$ -models

$x = (x^-, x^+) \in \mathbf{R}^2$  ... lightcone coordinates ,

$G$  ... n-dimensional Lie group,

$g : \mathbf{R}^2 \rightarrow G$  ... field

$y^\# : U(g) \rightarrow \mathbf{R}^n$  coordinate map on the neighbourhood of  $e \in G$ ,

$\phi : \mathbf{R}^2 \rightarrow \mathbf{R}^n$ ,  $\phi = y^\# \circ g$  ... field coordinates,

## $\sigma$ -model:

The **action** of the model can be written in two equivalent ways

$$\begin{aligned} S_F[\phi] &= \int dx^2 F_{ij}(\phi) \partial_+ \phi^i \partial_- \phi^j = \\ S_E(g) &= \int dx^2 E_{ab}(g) (\partial_- g g^{-1})^a (\partial_+ g g^{-1})^b, \end{aligned} \quad (1)$$

where  $F_{ij}(y) = e_i^a(\gamma) E_{ab}(\gamma) e_j^b(\gamma)$ ,  $\gamma \in G$ ,  $y^\#(\gamma) = y$  and  $e_i^a$  are components of vielbeins  $e_i^a(\gamma) = ((d\gamma)_i \cdot \gamma^{-1})^a$  in a given basis  $X_a$  of  $\mathfrak{g}$ .  $F_{ij}$  can be interpreted as a sum of metric and torsion potential in the target manifold  $G$ .

## Equations of motion

$$\partial_+ J_a^+ + \partial_- J_a^- = \mathcal{L}_{X_a}(F)_{ij} \partial_+ \phi^i \partial_- \phi^j, \quad a = 1, \dots, \dim G. \quad (2)$$

$\mathcal{L}_{X_a}$  are the Lie derivatives w.r.t left-invariant vector fields  $X_a$  where

$$[X_a, X_b] = f_{ab}^c X_c \quad (3)$$

and

$$J_a^\pm := F_{ij} X_a^i \partial_\mp \phi^j. \quad (4)$$

# Poisson–Lie T–duality of $\sigma$ –models

**C. Klimčík and P. Ševera**, Dual non–Abelian duality and the Drinfeld double, *Phys.Lett. B*, 351:455–462, 1995 [hep-th/9502122].

**C. Klimčík**, Poisson–Lie T-duality, *Nucl.Phys B (Proc.Suppl.)*, 46:116–121, 1996 [hep-th/9509095].

If there are  $\tilde{f}_a^{bc}$  – structure coefficients of a Lie group  $\tilde{G}$ ,  $\dim G = \dim \tilde{G}$  so that  $F$  satisfies

$$(\mathcal{L}_{X_c} F)_{ij} = \tilde{f}_c^{ab} X_a^m X_b^n F_{im} F_{nj} \quad (5)$$

for left–invariant fields  $X_a$  on  $G$  then there is  $\tilde{F}_{ij}$  on  $\tilde{G}$  and there is a relation between solutions of the equations of motion for  $S_F$  and  $S_{\tilde{F}}$ . These  $\sigma$ –models are called PLT–dual.

Necessary conditions: (5) and  $[\mathcal{L}_{X_a}, \mathcal{L}_{X_b}] = f_{ab}^c \mathcal{L}_{X_c} \Rightarrow$

$$f_{ab}^c \tilde{f}_c^{rs} + \tilde{f}_{[a}^{j[r} f_{b]j}^s] = 0 \quad (6)$$

These conditions are invariant with respect to  $f \leftrightarrow \tilde{f}$  so that we can solve (5) for both  $(f, \tilde{f}) \mapsto F$  and  $(\tilde{f}, f) \mapsto \tilde{F}$ .

(6)  $\Leftrightarrow$  Jacobi identities for the Drinfel’d double:

**Dual  $\sigma$ –models are naturally constructed on Drinfel’d doubles.**

## Drinfel'd doubles

$D$  – connected Lie group,  $\mathfrak{d}$  – corresponding Lie algebra,

$\langle , \rangle$  – bilinear form on  $\mathfrak{d}$ , symmetric, nondegenerate, ad–invariant:

$$ad_X \langle Y, Z \rangle = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \quad \forall X, Y, Z \in \mathfrak{d}$$

$D$  is the Drinfel'd double :  $\Leftrightarrow \exists \mathfrak{g}, \tilde{\mathfrak{g}} \subset \mathfrak{d}$

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subset \tilde{\mathfrak{g}}$$

$$\mathfrak{d} = \mathfrak{g} + \tilde{\mathfrak{g}}$$

$$\langle \mathfrak{g}, \mathfrak{g} \rangle = 0, \quad \langle \tilde{\mathfrak{g}}, \tilde{\mathfrak{g}} \rangle = 0$$

$\mathfrak{g}, \tilde{\mathfrak{g}}$  are maximal isotropic.

$(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  – Manin triple

$D$  is the Drinfel'd double  $\Rightarrow \dim \mathfrak{g} = \dim \tilde{\mathfrak{g}} \Rightarrow \dim \mathfrak{d} = \text{even}$ .

One can choose dual bases  $X_a \in \mathfrak{g}, \tilde{X}^b \in \tilde{\mathfrak{g}}$  so that  $\langle X_a, \tilde{X}^b \rangle = \delta_a^b$

$$[X_a, X_b] = f_{ab}^c X_c, \quad [\tilde{X}^a, \tilde{X}^b] = \tilde{f}^{ab}_c \tilde{X}^c$$

ad–invariance of  $\langle , \rangle \Rightarrow$

$$[X_a, \tilde{X}^b] = f_{ca}^b \tilde{X}^c + \tilde{f}_a^{bc} X_c$$

Jacobi identities  $\Leftrightarrow$  (6).

Due to Drinfel'd:  $\exists U(e) \subset D \quad \forall d \in U(e) \quad \exists g, h \in G, \tilde{g}, \tilde{h} \in \tilde{G}$

$$d = g \cdot \tilde{g} = \tilde{h} \cdot h$$

## Dual $\sigma$ -models

The equations of motion for  $\mathcal{S}_F[\phi] = \mathcal{S}_E[g]$  that satisfy the Klimčík – Ševera condition (5) can be written as equations on the whole Drinfel'd double:

$$\langle (\partial_{\pm} l) l^{-1}, \mathcal{E}^{\pm} \rangle = 0, \quad (7)$$

where

$$l = g \cdot \tilde{h} : \mathbb{R}^2 \rightarrow D, \quad g(x_+, x_-) \in \mathfrak{g}, \quad \tilde{h}(x_+, x_-) \in \tilde{\mathfrak{g}}$$

and subspaces are defined

$$\begin{aligned} \mathcal{E}^+ &= \text{span}(X_a + E_{ab}(e) \tilde{X}^b), \quad \mathcal{E}^- = \text{span}(X_a - E_{ba}(e) \tilde{X}^b), \\ \mathfrak{d} &= \mathcal{E}^+ + \mathcal{E}^-, \quad \langle \mathcal{E}^+, \mathcal{E}^- \rangle = 0 \end{aligned}$$

$\Rightarrow$  Dualizable models on  $G$  are given by the Manin triples  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  and (constant) matrices  $E(e)$ . We can decompose  $l$  as

$$l = \tilde{g} \cdot h \quad (8)$$

and obtain dual  $\sigma$ -model on  $\tilde{G}$ . The relation between the solutions  $g, \tilde{g}$  of the dual models follows from  $\tilde{g} \cdot h = g \cdot \tilde{h}$ .

## Plurality of $\sigma$ -models

It happens that  $D$  has more than two dual Manin triples  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  and  $(\mathfrak{d}, \tilde{\mathfrak{g}}, \mathfrak{g})$  i.e.

$$\mathfrak{d} = \mathfrak{g} + \tilde{\mathfrak{g}} = \mathfrak{b} + \tilde{\mathfrak{b}}, \quad \mathfrak{b} \neq \mathfrak{g}, \tilde{\mathfrak{g}}.$$

The possibility to decompose some Drinfel'd doubles into more than two Manin triples and the fact that (7) doesn't depend on the choice of the Manin triple enables to construct more than two equivalent sigma models on  $G, \tilde{G}, B, \tilde{B}, \dots$  from the decompositions

$$l = g \cdot \tilde{h} = \tilde{g} \cdot h = b \cdot \tilde{c} = \tilde{b} \cdot c = \dots$$

This property can be called **Poisson–Lie T-plurality**.

# Quantum $\sigma$ -models

In order that the  $\sigma$ -models can be properly quantized the action (1) is made conformal invariant . This requires an additional term containing the dilaton field  $\Phi(\mathbf{y})$  in the action  $S = S_{F,\Phi}$ .

**Conformal invariance**  $\Leftrightarrow$  vanishing  $\beta$ -functions.

In the first order of perturbation it requires that the fields  $F_{ij}$  and  $\Phi$  satisfy the equations

$$0 = R_{ij} - \nabla_i \nabla_j \Phi - \frac{1}{4} H_{imn} H_j^{mn} \quad (9)$$

$$0 = \nabla^k \Phi H_{kij} + \nabla^k H_{kij} \quad (10)$$

$$0 = R - 2 \nabla_k \nabla^k \Phi - \nabla_k \Phi \nabla^k \Phi - \frac{1}{12} H_{kmn} H^{kmn} \quad (11)$$

where the covariant derivatives  $\nabla_k$ , Ricci tensor  $R_{ij}$  and scalar curvature  $R$  are calculated from the metric

$$G_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) \quad (12)$$

that is also used for lowering and raising indices and

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \quad (13)$$

where the torsion potential is

$$B_{ij} = \frac{1}{2}(F_{ij} - F_{ji}). \quad (14)$$

# Transformation of conformally invariant $\sigma$ -models

R. von Unge, Poisson–Lie T–plurality, *JHEP*, 07:014, 2002 [hep-th/0205245].

$\{X_a, \tilde{X}^b\}$  – basis of the Manin triple  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ ,  $j, k \in \{1, \dots, n\}$ .

$\{U_a, \tilde{U}^b\}$  – basis of another Manin triple  $(\mathfrak{d}, \mathfrak{g}_u, \tilde{\mathfrak{g}}_u)$  of the same Drinfel’d double.

$$\begin{pmatrix} X \\ \tilde{X} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \quad (15)$$

The transformed model is then given by the action with  $\mathbf{E}$  replaced by

$$\mathbf{E}_u = M(N + \Pi_u M)^{-1} \quad (16)$$

and dilaton

$$\Phi_u = \Phi + \ln \text{Det}(N + \Pi_u M) - \ln \text{Det}(1 + \Pi E_0) + \ln \text{Det } \mathbf{a}_u - \ln \text{Det } \mathbf{a} \quad (17)$$

where

$$M = S^T E_0 - Q^T, \quad N = P^T - R^T E_0, \quad (18)$$

and  $\Pi, \mathbf{a}, \Pi_u, \mathbf{a}_u$  are calculated from the adjoint representation of the group  $\mathbf{G}$  and  $\mathbf{G}_u$  generated by  $X_a$  resp.  $U_a$  related by (15).

Unfortunately, the formula (17) is **not generally applicable** as it is a function on the whole Drinfel'd double but the sigma model must not depend on the auxiliary subgroup  $\tilde{\mathbf{G}}_u$ . This produces so called "dilaton puzzle" that was discussed in detail in

**L. Hlavatý, L. Šnobl**, Poisson–Lie T–plurality ..., *JHEP*, 0410(2004)045, [hep-th/0408126]

where the following conditions for the applicability of (17) are given.

**Theorem 1** *The dilaton (17) for the model defined on the group  $\mathbf{G}_u$  exists if and only if  $\forall g_u \in \mathbf{G}_u, \tilde{g}_u \in \tilde{\mathbf{G}}_u, \tilde{U} \in \tilde{\mathfrak{g}}_u$ ,*

$$\tilde{U}\Phi^{(0)}(g_u \cdot \tilde{g}_u) = \frac{d}{dt}\Phi^{(0)}(g_u \cdot \tilde{g}_u \cdot \exp(t\tilde{U}))|_{t=0} = 0,$$

where the function  $\Phi^{(0)}$

$$\Phi^{(0)} = \Phi - \ln|\text{Det}(1 + \Pi E_0)| - \ln|\text{Det } a| \quad (19)$$

is to be considered a function on the whole Drinfeld double  $\mathbf{D}$  by trivial extension

$$\Phi^{(0)}(g \cdot \tilde{g}) = \Phi^{(0)}(g), \quad (20)$$

and  $\tilde{U} \in \tilde{\mathfrak{g}}_u$  is extended as a left-invariant vector field on  $\mathbf{D}$ .

A simpler, more useful, necessary condition for the existence of the dilaton (17) for the model defined on the group  $\mathbf{G}_u$  is

$$\frac{d}{dt}\Phi^{(0)}(\exp(t\tilde{U}))|_{t=0} = 0, \quad \forall \tilde{U} \in \tilde{\mathfrak{g}}_u.$$

# Search for dual models – general strategy:

We are looking for pluralizable quantum  $\sigma$ -models. They are given by  $\mathbf{E}_{ij}$  and  $\Phi$  satisfying (5) and (9–11).

1. Take  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  and general matrix  $\mathbf{E}_{ij}(\mathbf{e}) \Rightarrow \sigma$ -model that satisfy the Klimčík–Ševera condition (5).  
The structure coefficients of  $\tilde{\mathfrak{g}}$  must be traceless  $\tilde{f}_b^{ab} = \mathbf{0}$ , otherwise there appears an anomaly that cannot be absorbed into the transformation of the dilaton.
2. Check vanishing  $\beta$ -function equations (9-11) with **constant dilaton**.  
 $\Rightarrow$  restrictions on  $\mathbf{E}_{ij}(\mathbf{e})$ .
3. Take all other Manin triples in the Drinfel'd double given by  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ .  
When the assumptions of Theorem 1 are satisfied construct transformed models and check vanishing of their  $\beta$ -function.

For computational reasons we restrict ourselves to 6-dimensional Drinfel'd doubles with solvable subalgebras  $\mathfrak{g}, \tilde{\mathfrak{g}}$ , giving sets of 3-dimensional Poisson–Lie T-plural models.

Alltogether there are 22 classes of 6-dimensional Drinfel'd doubles which contain 87 non-isomorphic classes of Manin triples.

**L. Hlavatý, L. Šnobl**, Classification of 6-dimensional Manin triples, [math.QA/0202209].

**L. Šnobl, L. Hlavatý**, Classification of 6-dimensional real Drinfel'd doubles, *Int.J.Mod.Phys. A17* (2002) 4043 [math.QA/0202210].

# Overview of results for $\dim \mathfrak{g} = 3$

**L. Hlavatý, L. Šnobl**, Poisson–Lie T–plurality of three–dimensional conformally invariant sigma models, *JHEP*, 0405(2004)010, [hep-th/0403164].

**L. Hlavatý, L. Šnobl**, Poisson–Lie T–plurality of three–dimensional conformally invariant sigma models II : Nondiagonal metrics and dilaton puzzle, *JHEP*, 0410(2004)045, [hep-th/0408126].

- We were able to identify the source of the dilaton puzzle as generally improper assumption of the independence of regularizing terms on the auxiliary group  $\tilde{G}$ .
- We wrote down conditions for applicability of the dilaton transformation formula (see Theorem 1).
- In addition to the well–known abelian case, we have found six types of  $\sigma$ –models on nonabelian groups for which the vanishing  $\beta$ –function equations hold for  $\Phi(\mathbf{y}) = \mathbf{0}$ .

Example 1:  $\sigma$ –model on  $\mathfrak{5}$  with

$$F(\mathbf{y}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & e^{-y_1} \\ \mathbf{0} & e^{-2y_1} & \mathbf{0} \\ e^{-y_1} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (21)$$

Example 2:  $\sigma$ –model on  $\mathfrak{6}_0$  with

$$F(\mathbf{y}) = \kappa \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{y}_2 \\ \mathbf{0} & -\mathbf{1} & -\mathbf{y}_1 \\ \mathbf{y}_2 & -\mathbf{y}_1 & 1 - \mathbf{y}_1^2 + \mathbf{y}_2^2 \end{pmatrix}. \quad (22)$$

Both these model are flat and torsionless.

- For each of the types we have found a set of mutually equivalent Poisson-Lie T-plural models with properties not encountered before in the context of Poisson-Lie T-plural models:

1. Models that allow two different dilatons

Example:  $\sigma$ -model on  $\mathbf{R}^3$  with

$$F(\mathbf{y}) = \Delta^{-1} \begin{pmatrix} \mathbf{y}_1^2 & \mathbf{y}_1(1 - \mathbf{y}_2) & 1 - \mathbf{y}_1 - \mathbf{y}_2 \\ -\mathbf{y}_1(1 + \mathbf{y}_2) & -1 + \mathbf{y}_2^2 & -1 + \mathbf{y}_1 + \mathbf{y}_2 \\ 1 - \mathbf{y}_1 + \mathbf{y}_2 & 1 - \mathbf{y}_1 + \mathbf{y}_2 & 0 \end{pmatrix} \quad (23)$$

where  $\Delta = (\mathbf{y}_1 - 1)^2 - \mathbf{y}_2^2$ . The  $\beta$ -equations hold both for  $\Phi = \ln|\Delta|$  and  $\Phi = \mathbf{const}$ . This model is PLT-equivalent to (22). It is flat and torsionless.

2. Equivalent models with the same Manin triple but different matrices  $\mathbf{E}(\mathbf{e})$ , one of them being flat with constant dilaton, the others being curved and with nontrivial dilaton.

Example :  $\sigma$ -models on  $\mathbf{R}^3$  with (23) and

$$F(\mathbf{y}) = (1 - \mathbf{y}_1^2 + \mathbf{y}_2^2)^{-1} \begin{pmatrix} 1 - \mathbf{y}_1^2 & \mathbf{y}_1 \mathbf{y}_2 & \mathbf{y}_2 \\ \mathbf{y}_1 \mathbf{y}_2 & -1 - \mathbf{y}_2^2 & -\mathbf{y}_1 \\ -\mathbf{y}_2 & \mathbf{y}_1 & 1 \end{pmatrix}, \quad (24)$$

$$\Phi(\mathbf{y}) = \ln |1 + \mathbf{y}_1^2 - \mathbf{y}_2^2|.$$

This model is curved with Gaussian curvature

$$R = \frac{10 - 4\mathbf{y}_1^2 + 4\mathbf{y}_2^2}{(1 + \mathbf{y}_1^2 - \mathbf{y}_2^2)^2}$$

and torsion  $H_{123} = 2(1 + \mathbf{y}_1^2 - \mathbf{y}_2^2)^{-2}$ .

3. Inequivalent models with the same Manin triple but different matrices  $\mathbf{E}(\mathbf{e})$ .

Example :  $\sigma$ -model on  $\mathfrak{6}_0$

$$F(\mathbf{y}) = \begin{pmatrix} 1 & 1 & \rho \\ 1 & 1 & 1 + \rho \\ \rho & 1 + \rho & 2\mathbf{y}_1 + \rho^2 \end{pmatrix}, \quad \rho = \mathbf{y}_1 + \mathbf{y}_2, \quad \Phi(\mathbf{y}) = 2\mathbf{y}_3. \quad (25)$$

This model model is PLT-equivalent to (21) but not to (22). It is curved with  $\mathbf{R} = \mathbf{0}$  and torsionless.

4. The curvatures of some models diverge on hypersurfaces where the corresponding dilatons are also divergent. Nevertheless, the metric, torsion potential and dilaton appear to have a reasonable continuation behind the singularity since all of them are well-defined. We don't know at the moment whether such backgrounds have meaningful physical interpretation, e.g. as branes.

Example: See (24)

## Open questions and future prospects

- One tends to believe that there should exist a description of the dilaton as some object on the whole Drinfel'd double not depending on the concrete choice of Manin triples, but none is known.
- All equivalence between models is currently only local, global issues are not at the present level of understanding covered by Poisson-Lie T-duality at all.
- We would like to express our opinion that in spite of the current, still very limited, understanding of quantum properties of Poisson-Lie T-duality it might already give us some practical results, e.g. can be used to generate rather nontrivial examples of conformally invariant string backgrounds. We hope that explicit examples can be used as useful nontrivial “guinea pigs” in future attempts to improve quantum version of Poisson-Lie T-duality and that the improved version of it might avoid some of the pitfalls encountered in the current investigation.