Week 8 (due May 27)

1. (a) Let $M$ be a Kähler manifold with Kähler metric $g_{i\bar{j}}$. Show that the only nonvanishing components of the Riemann tensor are

$$R_{i\bar{j}k\bar{l}} = -R_{jik\bar{l}} = -R_{i\bar{j}k\bar{l}} = R_{jik\bar{l}}.$$

Show that

$$R_{j\bar{k}l} = g^{\bar{m}i}R_{i\bar{j}k\bar{l}} = \partial_k\Gamma_{ji}^{m}.\tag{1}$$

(b) Show that the Ricci tensor on a Kähler manifold is of type $(1,1)$ and can be written as

$$R_{ji} = -\partial_j\partial_i \log \det g,$$

where $g$ is the matrix with entries $g_{k\bar{l}}$.

2. Show that locally on a Kähler manifold $M$ there exists a function $K$ (called the Kähler potential) such that

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}} K.$$

Show that $K$ cannot be globally-defined if $M$ is compact. Determine $K$ for $\mathbb{C}P^n$ with the Fubini-Study metric.

3. Combining the results of problems 1 and 2, we see that the Ricci tensor can be written as

$$R_{ji} = -\partial_i\partial_j \log \det \frac{\partial^2 K}{\partial z^k \partial z^l}.$$

Thus the condition of being Ricci-flat boils down to a nonlinear PDE for a single (but multi-valued) function $K$:

$$\partial_i\partial_j \log \det \frac{\partial^2 K}{\partial z^k \partial z^l} = 0.$$

Such a PDE is called a Monge-Ampère-type equation.

(b) A Riemannian manifold is called an Einstein manifold if the Ricci tensor satisfies

$$R_{\mu\nu} = \lambda g_{\mu\nu},$$

where $\lambda$ is a real number (essentially, $\lambda$ is the cosmological constant). Compute the Ricci tensor for $\mathbb{C}P^n$ with the Fubini-Study metric and show that it is an Einstein manifold.