Week 7 (due May 20)

1. Let $h$ be a Hermitian metric on a complex manifold $M$, let $g$ be the corresponding Riemannian metric, and $\omega$ the corresponding 2-form. Let $J$ be the integrable almost complex structure tensor on $M$ corresponding to its complex structure. We can regard $g$ and $\omega$ as bundle maps from $TM$ to $TM^*$, while $J$ is a map from $TM$ to $TM$. These three maps are algebraically related; find this relationship.

2. Show that $h$ defines a Kähler structure on $M$ if and only if the tensor $J$ is covariantly constant with respect to the Levi-Civita connection corresponding to the metric $g$.

3. Let $X$ be a real vector field on a complex manifold $M$. It can be decomposed into $(1,0)$ and $(0,1)$ parts which are complex vector fields on $M$. One says that $X^{1,0}$ is holomorphic if its components in holomorphic coordinates are holomorphic functions; it is easy to see that this definition does not depend on the choice of the holomorphic coordinate system. Show that $X^{1,0}$ is holomorphic if and only if the Lie derivative of $J$ with respect to $X$ vanishes. Here $J$ is as above.

4. Let $N$ be a (not necessarily complex) submanifold of a complex manifold $M$. Show that $N$ is a complex submanifold if and only if $TN \subset TM$ is preserved by the complex structure tensor $J : TM \rightarrow TM$. 

1