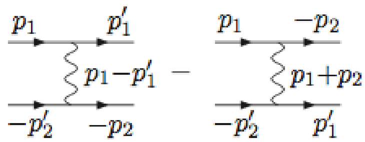
1. Srednicki 59.2

59.2) For $e^+e^- \to e^+e^-$, the diagrams are



and the amplitude is

$$\mathcal{T} = e^2\left[\frac{(\overline{u}_1'\gamma^\mu u_1)(\overline{v}_2\gamma_\mu v_2')}{-t} - \frac{(\overline{v}_2\gamma^\mu u_1)(\overline{u}_1'\gamma_\mu v_2')}{-s}\right]. \tag{59.44}$$

We then have

$$\overline{\mathcal{T}} = e^2\left[\frac{(\overline{u}_1\gamma^\nu u_1')(\overline{v}_2'\gamma_\nu v_2)}{-t} - \frac{(\overline{u}_1\gamma^\nu v_2)(\overline{v}_2'\gamma_\nu u_1')}{-s}\right]. \tag{59.45}$$

Thus

$$|\mathcal{T}|^2 = e^4\left[\Phi_{tt}/t^2 + (\Phi_{ts}+\Phi_{st})/ts + \Phi_{ss}/s^2\right], \tag{59.46}$$

where

$$\begin{aligned}
\Phi_{tt} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(u_1'\overline{u}_1')\gamma^\mu]\,\mathrm{Tr}[(v_2\overline{v}_2)\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu]\,,\\
\Phi_{ss} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(v_2\overline{v}_2)\gamma^\mu]\,\mathrm{Tr}[(u_1'\overline{u}_1')\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu]\,,\\
\Phi_{ts} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(v_2\overline{v}_2)\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu(u_1'\overline{u}_1')\gamma^\mu]\,,\\
\Phi_{st} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(u_1'\overline{u}_1')\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu(v_2\overline{v}_2)\gamma^\mu]\,.
\end{aligned} \tag{59.47}$$

Averaging over initial spins and summing over final spins yields

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac14 \mathrm{Tr}[(-\not p_1+m)\gamma^\nu(-\not p_1'+m)\gamma^\mu]\,\mathrm{Tr}[(-\not p_2-m)\gamma_\mu(-\not p_2'-m)\gamma_\nu]\;,\\
\langle\Phi_{ss}\rangle &= \tfrac14 \mathrm{Tr}[(-\not p_1+m)\gamma^\nu(-\not p_2-m)\gamma^\mu]\,\mathrm{Tr}[(-\not p_1'+m)\gamma_\mu(-\not p_2'-m)\gamma_\nu]\;,\\
\langle\Phi_{ts}\rangle &= \tfrac14 \mathrm{Tr}[(-\not p_1+m)\gamma^\nu(-\not p_2-m)\gamma_\mu(-\not p_2'-m)\gamma_\nu(-\not p_1'+m)\gamma^\mu]\;,\\
\langle\Phi_{st}\rangle &= \tfrac14 \mathrm{Tr}[(-\not p_1+m)\gamma^\nu(-\not p_1'+m)\gamma_\mu(-\not p_2'-m)\gamma_\nu(-\not p_2-m)\gamma^\mu]\;.
\end{aligned}\tag{59.48}$$

We see that exchanging $p_1' \leftrightarrow -p_2$, which is equivalent to $t \leftrightarrow s$, exchanges $\langle\Phi_{tt}\rangle \leftrightarrow \langle\Phi_{ss}\rangle$ and $\langle\Phi_{ts}\rangle \leftrightarrow \langle\Phi_{st}\rangle$. We have

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac14\Big(\mathrm{Tr}[\not p_1\gamma^\nu\not p_1'\gamma^\mu] + m^2\,\mathrm{Tr}[\gamma^\nu\gamma^\mu]\Big)\Big(\mathrm{Tr}[\not p_2\gamma_\mu\not p_2'\gamma_\nu] + m^2\,\mathrm{Tr}[\gamma_\mu\gamma_\nu]\Big)\\
&= 4\Big(p_1^\nu p_1'^{\,\mu} + p_1^\mu p_1'^{\,\nu} - (p_1p_1' + m^2)g^{\mu\nu}\Big)\Big(p_{2\mu}p_{2\nu}' + p_{2\nu}p_{2\mu}' - (p_2p_2' + m^2)g_{\nu\mu}\Big)\\
&= 4[2(p_1p_2')(p_1'p_2) + 2(p_1p_2)(p_1'p_2') - 2(p_1p_1')(p_2p_2' + m^2) - 2(p_2p_2')(p_1p_1' + m^2)\\
&\qquad + 4(p_1p_1' + m^2)(p_2p_2' + m^2)]\;.
\end{aligned}\tag{59.49}$$

Now we use

$$\begin{aligned}
p_1p_2 = p_1'p_2' &= -\tfrac12(s - 2m^2)\;,\\
p_1p_1' = p_2p_2' &= +\tfrac12(t - 2m^2)\;,\\
p_1p_2' = p_1'p_2 &= +\tfrac12(u - 2m^2) = \tfrac12(2m^2 - s - t)
\end{aligned}\tag{59.50}$$

and simplify to get

$$\langle\Phi_{tt}\rangle = 2(t^2 + 2st + 2s^2 - 8m^2s + 8m^4)\;. \tag{59.51}$$

Swapping $t \leftrightarrow s$ yields

$$\langle\Phi_{ss}\rangle = 2(s^2 + 2st + 2t^2 - 8m^2t + 8m^4)\;. \tag{59.52}$$

Next we have

$$\begin{aligned}
\langle\Phi_{ts}\rangle = &\ \tfrac14 \mathrm{Tr}[\not p_1\gamma^\nu\not p_2\gamma_\mu\not p_2'\gamma_\nu\not p_1'\gamma^\mu]\\
&- \tfrac14 m^2\,\mathrm{Tr}[\not p_1\gamma^\nu\not p_2\gamma_\mu\gamma_\nu\gamma^\mu]\\
&- \tfrac14 m^2\,\mathrm{Tr}[\not p_1\gamma^\nu\gamma_\mu\not p_2'\gamma_\nu\gamma^\mu]\\
&+ \tfrac14 m^2\,\mathrm{Tr}[\not p_1\gamma^\nu\gamma_\mu\gamma_\nu\not p_1'\gamma^\mu]\\
&+ \tfrac14 m^2\,\mathrm{Tr}[\gamma^\nu\not p_2\gamma_\mu\not p_2'\gamma_\nu\gamma^\mu]\\
&- \tfrac14 m^2\,\mathrm{Tr}[\gamma^\nu\not p_2\gamma_\mu\gamma_\nu\not p_1'\gamma^\mu]\\
&- \tfrac14 m^2\,\mathrm{Tr}[\gamma^\nu\gamma_\mu\not p_2'\gamma_\nu\not p_1'\gamma^\mu]\\
&+ \tfrac14 m^4\,\mathrm{Tr}[\gamma^\nu\gamma_\mu\gamma_\nu\gamma^\mu]\;.
\end{aligned}\tag{59.53}$$

In the first line, $\not p_1\gamma^\nu\not p_2\gamma_\mu\not p_2'\gamma_\nu\not p_1'\gamma^\mu = 2\not p_1\not p_2'\gamma_\mu\not p_2\not p_1'\gamma^\mu = (2\not p_1\not p_2')(4p_2p_1')$. In the second line, $\not p_1\gamma^\nu\not p_2\gamma_\mu\gamma_\nu\gamma^\mu = 4\not p_1 p_{2\mu}\gamma^\mu = 4\not p_1\not p_2$; the next five lines can be similarly simplified. In the last line, $\gamma^\nu\gamma_\mu\gamma_\nu\gamma^\mu = 2\gamma_\mu\gamma^\mu = -8$. Taking traces yields

$$\langle\Phi_{ts}\rangle = -8(p_1p_2')(p_1'p_2) + 4m^2(p_1p_2 + p_1p_2' - p_1p_1' - p_2p_2' + p_1'p_2 + p_1'p_2') - 8m^4\;, \tag{59.54}$$

and plugging in eq. (59.50), we find

$$\langle \Phi_{ts} \rangle = -2(u^2 - 8m^2 u + 12m^4) \ . \tag{59.55}$$

Swapping $t \leftrightarrow s$, we get

$$\langle \Phi_{st} \rangle = -2(u^2 - 8m^2 u + 12m^4) \ . \tag{59.56}$$

2. Srednicki 62.2

62.2) Only the photon propagator is changed. Since the one-loop contribution to $\Pi^{\mu\nu}(k)$ does not include a photon propagator, Z_3 is unchanged at one loop. The extra term $(\xi-1)k^\mu k^\nu/(k^2)^2$ in the photon propagator would add an extra term to the electron self-energy of the form

$$i\Delta\Sigma(p) = (\xi-1)e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\not{\ell}(-\not{p}-\not{\ell}+m)\not{\ell}}{((p+\ell)^2+m^2)(\ell^2)^2} - i(\Delta Z_2)\not{p} - i(\Delta Z_m)m \,, \tag{62.51}$$

where ΔZ_2 and ΔZ_m are the extra contributions to Z_2 and Z_m that are needed to cancel the extra contributions to the divergence. Combining denominators with Feynman's formula yields

$$\begin{aligned} \frac{1}{((p+l)^2+m^2)(l^2)^2} &= \int dF_3\,[x_1(p+\ell)^2 + x_1 m^2 + x_2\ell^2 + x_3\ell^2]^{-3} \\ &= \int dF_3\,[\ell^2 + 2x_1\ell\cdot p + x_1 p^2 + x_1 m^2]^{-3} \\ &= \int dF_3\,[(\ell+x_1 p)^2 + x_1(1-x_1)p^2 + x_1 m^2]^{-3} \\ &= 2\int_0^1 dx\,(1-x)[q^2+D]^{-3} \,, \end{aligned} \tag{62.52}$$

where $q = \ell + xp$ and $D = x(1-x)p^2 + xm^2$. We set $\ell = q - xp$ in the numerator, and drop terms that are odd in q. Then only the q^2 terms contribute to the divergence. These terms are $x(\not{q}\not{q}\not{p} + \not{q}\not{p}\not{q} + \not{p}\not{q}\not{q}) + \not{q}(-\not{p}+m)\not{q}$, and making the replacement $q^\mu q^\nu \to \frac{1}{4}q^2 g^{\mu\nu}$ yields $\frac{1}{4}q^2[x(\gamma^\mu\gamma_\mu\not{p} + \gamma^\mu\not{p}\gamma_\mu + \not{p}\gamma^\mu\gamma_\mu) + \gamma^\mu(-\not{p}+m)\gamma_\mu] = \frac{1}{4}q^2[x(-4\not{p}+2\not{p}-4\not{p}) - 2\not{p} - 4m] = -\frac{1}{4}q^2[(6x+2)\not{p}+4m]$. (We can set $d=4$ because terms of order ε will not contribute to the divergent part.) Then we use

$$\int \frac{d^4q}{(2\pi)^4} \frac{q^2}{(q^2+D)^3} = \frac{i}{8\pi^2\varepsilon} + \text{finite} \tag{62.53}$$

and $2\int_0^1 dx\,(1-x) = 1$ and $2\int_0^1 dx\,(1-x)x = \frac{1}{3}$ to get

$$\Delta\Sigma(p) = -(\xi-1)\frac{e^2}{8\pi^2\varepsilon}(\not{p}+m) - (\Delta Z_2)\not{p} - (\Delta Z_m)m \,, \tag{62.54}$$

Combining this with eqs. (62.34) and (62.35), we get

$$Z_2 = 1 - \xi\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \,, \tag{62.55}$$

$$Z_m = 1 - (3+\xi)\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \,. \tag{62.56}$$

To compute the change in Z_1, we can set external momenta to zero. Then we have

$$i\Delta V^\mu(0,0) = ie\Delta Z_1\gamma^\mu + (\xi{-}1)e^3\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\not\ell(-\not\ell{+}m)\gamma^\mu(-\not\ell{+}m)\not\ell}{(\ell^2+m^2)^2(\ell^2)^2}\,. \tag{62.57}$$

Only the ℓ^4 term in the numerator gives a divergence, so we can replace the numerator with $\not\ell\not\ell\gamma^\mu\not\ell\not\ell = (\ell^2)^2\gamma^\mu$. The divergent part of the integral is then $i/8\pi^2\varepsilon$, and so the divergent part of ΔZ_1 is $-(\xi{-}1)e^2/8\pi^2\varepsilon$, leading to

$$Z_1 = 1 - \xi\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4)\,. \tag{62.58}$$

We see that $Z_1 = Z_2$ for all ξ, and that $Z_1 = Z_2 = 1 + O(e^4)$ for Lorenz gauge ($\xi = 0$). This will prove very convenient later.