1 Content of the course

"Quantum Field Theory" by M. Srednicki, Part 1.

2 Combining QM and relativity

We are going to keep all axioms of QM:

1. states are vectors (or rather rays) in Hilbert space.
2. observables are Hermitian operators and their values are the spectrum.
3. probability of measuring a particular value \(a\) of an observable \(A\) in a state \(\Psi\) is
\[
\frac{|\langle P_a \Psi \rangle|^2}{|\langle \Psi \rangle|^2},
\]
where \(P_a\) is a projector to the eigenspace of \(A\) corresponding to \(a\).
4. Time evolution of states is governed by the Schrödinger equation
\[
i \frac{d\Psi(t)}{dt} = H \Psi(t),
\]
where \(H\) is the Hamiltonian (energy operator).
5. Symmetries are unitary or anti-unitary operators preserving the Hamiltonian.
6. etc.

For a nonrelativistic particle, we let \(H = L^2(\mathbb{R}^3)\) and let the momentum operator (generator of translations) be \(\hat{P} = -i\hbar \nabla\). Since \(E = P^2/2m\) classically, it is natural to define \(H = \hat{P}^2/2m = -\hbar^2 \nabla^2/2m\).

From now on, I will let \(\hbar = 1\), so \(H = -\nabla^2/2m\).

For a relativistic particle,
\[
E = \sqrt{\hat{P}^2 c^2 + m^2 c^4},
\]
so can try
\[
H = \sqrt{-\nabla^2 c^2 + m^2 c^4}
\]
This expression is problematic: treats time and space asymmetrically and appears nonlocal.

Alternatively, we can “quantize” the squared dispersion relation \(E^2 = P^2 c^4 + m^2 c^4\) to get the Klein-Gordon equation
\[
\frac{\partial^2 \Psi}{\partial t^2} = (-c^2 \nabla^2 + m^2 c^4) \Psi.
\]
This equation is more reasonable, as it is more symmetric w.r. to exchange of time and space. To see relativistic invariance better, let \( x^0 = ct \). From now I will let \( c = 1 \), so in such units \( x^0 = t \). Also, \( x_0 = -t \), and \( x^i = x_i, i = 1, 2, 3 \). Greek indices will run over the set \( 0, 1, 2, 3 \).

Minkowski metric: \( g_{\mu\nu} = \text{diag}(-1,1,1,1) = g^{\mu\nu} \). \( x_\mu = g_{\mu\nu} x^\nu \), where we use Einstein’s convention (summation over repeated indices). Similarly, \( x^\mu = g^{\mu\nu} x_\nu \).

Minkowski interval: \( x^2 = x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} = -(x^0)^2 + \sum_i (x^i)^2 \).

Lorenz transformations are

\[
\bar{x}^\mu = \Lambda^\mu_\nu x^\nu,
\]

where \( \Lambda \) is any real matrix such that \( \bar{x}^\mu \bar{x}_\mu = x^\mu x_\mu \).

Now we can check relativistic invariance of the KG equation, i.e. that \( \phi(x) \) and \( \phi(\bar{x}) \) satisfy the same equation.

Let

\[
\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial^\mu = g^{\mu\nu} \partial_\nu.
\]

Then

\[
\bar{\partial}^\mu = \Lambda^\rho_\mu \partial^\rho.
\]

and therefore \( \bar{\partial}^2 = \partial^2 \). Hence the KG operator is Lorenz-invariant.

Problems:

1. \( \int d^3x |\Psi|^2 \) is not conserved. Moreover, it has wrong transformation properties under the Lorenz transformation: \( |\Psi|^2 \) is not a time component of a 4-vector, so we do not expect a continuity equation to hold (and it does not). One can write down something which is a component of a conserved 4-vector:

\[
j_\mu = i (\Psi^* \partial_\mu \Psi - \partial_\mu \Psi^* \Psi)
\]

satisfies \( \partial_\mu j^\mu = 0 \), and so

\[
\int d^3x j^0
\]

is conserved. But \( j^0 \) is not positive-definite, so cannot be interpreted as probability density.

2. Negative-energy solutions.

Dirac tried to solve these problems by looking for a first-order equation, but for a multicomponent wavefunction. This solved problem 1, but not problem 2.
Ultimately, the problem is that relativistic QM can be consistently developed only if we do not work in a theory with a fixed number of particles. Hence we need to understand how to describe systems where particle creation and annihilation is allowed.

3 Fock space methods (second quantization)

3.1 Bosons

A single particle has $\mathcal{H}_1 = L^2(\mathbb{R}^3)$ as its Hilbert space. Two particles have $\mathcal{H}_2 = L^2(\mathbb{R}^3 \times \mathbb{R}^3)_{\text{sym}} \simeq \text{Sym}^2 \mathcal{H}_1$. And so on. The Hilbert space without any particles is one-dimensional (the vacuum state). Thus

$$H = \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots = \bigoplus_{k=0}^{\infty} \text{Sym}^k(\mathcal{H}_1).$$

This is called the bosonic Fock space $\mathcal{F}(\mathcal{H}_1)$ associated to $\mathcal{H}_1$.

The Fock space is always infinite-dimensional, even if $\mathcal{H}_1$ is not. Let us look at the extreme case, $\mathcal{H}_1 \simeq \mathbb{C}$. Then

$$\mathcal{F}(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \ldots.$$ 

Thus a vector in $\mathcal{F}(\mathbb{C})$ is an infinite sequence of numbers or vector $(a_0, a_1, a_2, \ldots)$ such that $\sum_k |a_k|^2 < \infty$.

It is often convenient to think of such a sequence as Taylor coefficients of an analytic function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

Degree is then identified with the particle number. Polynomials form a dense set in this space of functions and correspond to states with a finite number of particles.

Two natural operations on polynomials are $z$ and $\partial$. They satisfy

$$[\partial, z] = 1.$$ 

One calls $\partial$ the annihilation operator $a$, and calls $z$ the creation operator $a^\dagger$. They are indeed conjugate to each other if we define the scalar product to be

$$||f(z)||^2 = \frac{1}{2\pi} \int d^2z |f(z)|^2 e^{-|z|^2}.$$
Using this scalar product, one can compute $\|z^n\|^2 = n!$. Thus a normalized $n$-particle state is

$$|n\rangle = \frac{1}{\sqrt{n!}} z^n = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle.$$  

Thus

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle.$$  

This can serve as a definition of creation and annihilation operators.

The particle number operator can be expressed as $N = z \partial_z = a^\dagger a$. Eigenstates of $N$ are homogeneous polynomials. Polynomials are Fock space states which involve only a finite number of particles.

More generally, suppose $\mathcal{H}_1 \simeq \mathbb{C}^N$. Let us choose a basis $\psi_i, i = 1, \ldots, N$ in $\mathcal{H}_1$ and introduce $N$ variables $z_1, \ldots, z_N$. Then one can identify $\text{Sym}^p(\mathcal{H}_1)$ with the space of polynomials in $N$ variables of total degree $p$: a state with $k_1$ particles in the state $\psi_1$, $k_2$ particles in the state $\psi_2$, etc. can be identified with the polynomial

$$z_1^{k_1} \cdots z_N^{k_N}.$$  

We define $a_i = \partial_i, a_i^\dagger = z_i$ so that

$$[a_i, a_j^\dagger] = \delta_{ij}.$$  

The Fock space is then the space of all polynomials in variables $z_1, \ldots, z_N$.

If we change the basis in $\mathcal{H}_1$, creation-annihilation operators also change: if $\psi'_i = B_i^j \psi_j$, where $B$ is a unitary matrix, then

$$a'_i = B_i^j a_j.$$  

The particle number operator is $N = \sum_i z_i \partial_i = \sum_i a_i^\dagger a_i$. Eigenstates of $N$ are homogenous polynomials.

If $\mathcal{H}_1$ is infinite-dimensional, but has a countable basis, we can still think of its Fock space as a completion the space of polynomials in variables $z_1, z_2, \ldots$.

But usual bases on $L^2(\mathbb{R}^3)$ (momentum eigenstates $|p\rangle$ and coordinate eigenstates $|x\rangle$) are not like that. Still, one can define analogues of creation and annihilation operators:

$$\Psi(x) = \sum_i a_i \psi_i(x), \quad \Psi^\dagger(x) = \sum_i a_i^\dagger \psi_i^\dagger(x).$$
They satisfy
\[ [\Psi(x), \Psi^\dagger(y)] = \delta^3(x - y). \]

All operators in Fock space can be expressed in terms of \( \Psi(x) \) and \( \Psi^\dagger(x) \).

Examples:
0. The particle number operator \( N = \int d^3x \Psi^\dagger(x)\Psi(x) \).
1. One-particle operators. A one-particle operator is an operator of the form
\[
\sum_{k=1}^{\infty} \sum_{i=1}^{k} 1 \otimes \ldots \otimes 1 \otimes O \otimes 1 \otimes \ldots \otimes 1 = \sum_{k=0}^{\infty} \sum_{i=1}^{k} O_i,
\]
where \( O \) is an operator on \( \mathcal{H}_1 \). It can be written as
\[
\int d^3x d^3y \Psi^\dagger(x) \langle x|O|y \rangle \Psi(y).
\]

For example, the kinetic energy operator is a one-particle operator with \( O = -\nabla^2/2m \), so the corresponding operator in Fock space is
\[
\int d^3x \Psi^\dagger(x) \left(-\nabla^2/2m\right) \Psi(x).
\]

The particle number operator is a one-particle operator with \( O = 1 \).

2. Two-particle operators. These are operators of the form
\[
\sum_{k=1}^{\infty} \sum_{1 \leq i < j \leq k} O_{ij}
\]
where \( O_{ij} \) is an operator on \( \mathcal{H}_2 \) which acts only on the \( i \)-th and \( j \)-th particle.

The corresponding operator in Fock space is
\[
\frac{1}{2} \int d^3x d^3y d^3zd^3t \Psi^\dagger(x)\Psi^\dagger(y)\langle x, y|O|z, t \rangle \Psi(t)\Psi(z).
\]

For example, the potential energy operator is of this form, with \( \langle x, y|O|z, t \rangle = V(x - y)\delta^3(x - z)\delta^3(y - t) \). The corresponding operator in Fock space is
\[
\frac{1}{2} \int d^3x d^3y \Psi^\dagger(x)\Psi^\dagger(y)V(x - y)\Psi(y)\Psi(x).
\]

How do we formulate dynamics in the Fock space? Since the emphasis is on creation-annihilation operators, it is often convenient to work in the
Heisenberg picture and write EOMs for $\Psi$ and $\Psi^\dagger$, instead of the Schrödinger equation. For free particles, we get

$$\frac{\partial \Psi}{\partial t} = i[H, \Psi] = -\frac{i}{2m} \nabla^2 \Psi.$$ 

This looks like Schrödinger equation, but for a field operator. Hence the name "second quantization". Let us find a solution. Go to momentum space:

$$\Psi(x) = \int d^3p (2\pi)^{-3} b(p) e^{ipx}$$

Then

$$[b(p), b^\dagger(q)] = (2\pi)^3 \delta^3(p - q).$$

and

$$H = \int d^3p (2\pi)^{-3} \frac{p^2}{2m} b^\dagger(p) b(p).$$

$$b(p, t) = e^{-iE_p t} b(p, 0).$$

Thus

$$\Psi(t, x) = \int \frac{d^3p}{(2\pi)^3} b(p, 0) e^{-E_p t + ipx}.$$ 

This completely determines the evolution of all observables.

For an interacting system, get the following equation:

$$i \frac{\partial \Psi}{\partial t} = -\frac{1}{2m} \nabla^2 \Psi + \int d^3y \Psi^\dagger(y) \Psi(y) V(x - y) \Psi(x).$$

This is nonlinear and cannot be regarded as "second-quantized" Schrödinger equation. Its classical analogue is a PDE for an ordinary function $\Psi(t, x)$, which is NOT interpreted as a quantum-mechanical wavefunction.

Remark: the quantum harmonic oscillator corresponds to the Fock space for $\mathcal{H} = 1$. A collection of $N$ harmonic oscillators is equivalent to the bosonic Fock space for $\mathcal{H}_1 = \mathbb{C}^N$. Thus the quantization of a system of harmonic oscillators can be interpreted in terms of free bosonic particles. The energies of 1-particle states are $\omega_i$. 
3.2 Fermions

Now consider fermionic particles which obey the Pauli principle. Fermionic wavefunctions are antisymmetric with respect to the exchange of any two particles.

Let us again begin with the case $\mathcal{H}_1 = \mathbb{C}$. Then the Fock space is

$$\mathcal{F}(\mathcal{H}_1) = \mathbb{C} \oplus \mathbb{C}.$$  

This is two-dimensional, and there are many ways to think about it. E.g., we can identify it with the states of a spin-1/2 particle. But we will choose a more esoteric viewpoint. Consider a variable $\theta$ which has a multiplication rule $\theta^2 = 0$. Then “analytic functions of $\theta$” are linear functions

$$f(\theta) = a + b\theta.$$  

The space of such functions can be identified with $\mathcal{F}(\mathcal{H}_1)$: th vacuum state is 1, while the 1-particle state is $\theta$.

Creation-annihilation operators are defined as before: $c^\dagger = \theta$, $c = \partial_\theta$. Note that $c^2 = (c^\dagger)^2 = 0$. It is also easy to check that $cc^\dagger + c^\dagger c = 1$. Note the crucial plus sign. The particle number operator is $N = c^\dagger c$.

We can define the scalar product so that $c^\dagger$ is indeed the adjoint of $c$. Details of this are left as an exercise.

Now consider $N$-dimensional $\mathcal{H}_1$. Introduce $N$ variables $\theta_i$ which satisfy $\theta_i \theta_j + \theta_j \theta_i = 0$. Consider an analytic function of $\theta_i$. Again, the series terminates in degree $N$. The total dimension of the space of functions is $2^N$. The $k$-ht term in the expansion is

$$\sum_{i_1 \ldots i_k} f^{i_1 \ldots i_k} \theta_{i_1} \ldots \theta_{i_k}.$$  

Here the coefficient functions are totally anti-symmetric, as required by the Fermi statistics. The creation operators are $c_i^\dagger = \theta_i$, the annihilation operators are $c_i = \partial_{\theta_i}$. They satisfy

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}.$$  

Note that the fermionic Fock space has a symmetry which replaces the vacuum with the “filled state” $\theta_1 \ldots \theta_N$ and exchanges $c_i$ and $c_i^\dagger$. There is nothing analogous in the bosonic case.
The rest proceeds as before. We can choose a countable basis in \( L^2(\mathbb{R}^3) \) and define

\[
\Psi(x) = \sum_i \psi_i(x) c_i, \quad \Psi^\dagger(x) = \sum_i \psi_i^\ast(x) c_i^\dagger.
\]

They satisfy

\[
\Psi(x)\Psi^\dagger(y) + \Psi^\dagger(y)\Psi(x) = \delta^3(x - y).
\]

These are called canonical anti-commutation relations. In the noninteracting case, the EOM is linear and solved exactly as in the bosonic case.

4 Classical field theory

There is something special about differential equations which come from “de-quantizing” the Heisenberg equations of motion: they come from a variational principle.

4.1 Classical mechanics

Recall classical mechanics. Action:

\[
S = \int_0^T dt L(q^i(t), \dot{q}^i(t)).
\]

Euler-Lagrange variational principle: \( \delta S = 0 \) with \( q(0) \) and \( q(T) \) fixed. Equations of motion:

\[
\frac{\partial L}{\partial q^i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right).
\]

Alternatively, we can introduce \( p_i = \partial L/\partial q^i \), the Hamiltonian

\[
H = p\dot{q} - L,
\]

and write the action as

\[
S = \int dt (p\dot{q} - H(p(t), q(t))).
\]

The equation \( \delta S = 0 \) then gives

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.
\]
These are Hamilton equations.

Finally, if we introduce the Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i}$$

for any two functions $F, G$, the Hamilton equations of motion can be written as

$$\dot{q}^i = \{H, q^i\}, \quad \dot{p}_i = \{H, p_i\}.$$  

We also have

$$\{p_i, q_j\} = \delta^j_i, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0.$$  

Under quantization, Poisson bracket becomes $\hbar$ times the commutator.

4.2 Nonrelativistic field theory

Now we want to have a similar formalism where $i$ is replaced with a continuous index $x$. Instead of $q_i(t)$ will have $\Psi(t, x)$. Action:

$$S = \int dt L(\Psi, \dot{\Psi}).$$

EOM:

$$\frac{\delta L}{\delta \Psi(t, x)} = \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\Psi}(t, x)} \right).$$

Here the variational derivative is defined by

$$\delta L = \int d^3x \frac{\delta L}{\delta \Psi(t, x)} \delta \Psi(t, x).$$

In the free case, it is sufficient to take

$$L = L_0 = \int d^3x \left( i\dot{\Psi}^* \dot{\Psi} - \frac{1}{2m} \partial_i \Psi^* \partial_i \Psi \right).$$

Note that $L$ is an integral of a local expression, $L = \int d^3x \mathcal{L}$, so

$$S = \int dt d^3x \mathcal{L}(\Psi, \dot{\Psi}).$$
This is very nice, but is not obligatory in a nonrelativistic situation: in an interacting case one finds

\[ L = L_0 - \frac{1}{2} \int d^3xd^3y|\Psi(x)|^2|\Psi(y)|^2V(x - y). \]

This is local in some very special cases. For example, when \( V(x) = \delta^3(x) \) ("contact interaction"). In the relativistic case only such interaction are allowed.

Note that this fits better with the second version of the variational principle: \( i\Psi^* \) is the "momentum conjugate to \( \Psi \)." So one has Poisson brackets

\[ \{\Psi^*(x), \Psi(y)\} = -i\delta^3(x - y). \]

The Hamiltonian is then given by the same expression as before, but \( \Psi \)'s are now ordinary functions, not Fock-space operators.

Quantization now is easy: we get the standard commutation relations for \( \Psi \) and \( \Psi^* \) and realize them as operators in Fock space.

How do we get fermionic Fock space in this way? There is no good way of doing so. Reason: classical limit makes sense only when a large number of particles are in the same state.

For clarity, consider discrete case. In order for the commutator term to be negligible, need to consider a state where \( a \) has a large expectation value (and small variance). Hence \( N = a^\dagger a \) will have a large expectation value. This is not possible in the fermionic case.

Formally, we can still consider the same equations, but with \( \Psi \) and \( \Psi^* \) satisfying anticommutation relations. This means that they are not ordinary functions, but generators of a Grassmann algebra. We will use this trick later.

### 4.3 Relativistic field theory

Main idea: interpret the KG equation not as an equation for a wavefunction, but an equation for a field operator. That is, let us make relativistic not the one-particle Schrodinger equation, but the Heisenberg equation of motion for the Fock space operator.

To understand it, we need to specify commutation relations for \( \Psi \) in such a way, that the KG equation is the Heisenberg equation for some Hamiltonian. We can do this like this: first solve an analogous classical problem, and then quantize everything.
The classical KG equation comes from the action

\[
S = \frac{1}{2} \int dt d^3x \left( -\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right).
\]

This looks more like the first version of the variational principle. The momentum is

\[
p(x) = \dot{\phi}(x),
\]

and the Hamiltonian is

\[
H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x \left( p(x)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right)
\]

The Poisson brackets are

\[
\{p(x), \phi(y)\} = \delta^3(x - y).
\]

Hence quantization will give

\[
[\phi(x), p(y)] = i \delta^3(x - y).
\]

This is just like \([q, p] = i\), but with continuous indices.

Reason: the classical system describes the continuum limit of a system of particles connected with springs, and \(\phi(x)\) is the continuum limit of the coordinate of a particle.

Classical excitations are waves. What about quantization? Expect that we get a system of free bosonic particles with a relativistic dispersion law. Two reasons: (1) that is what we set out to describe; (2) classical system can be Fourier-analyzed into a collection of harmonic oscillators; each oscillator is equivalent to a Fock space (for a one-dimensional vector space), so the whole thing is equivalent to a Fock space (for an infinite-dimensional 1-particle space), so describes free bosonic particles.