Problem 38.2

Denoting \( \beta A^\dagger \beta \), consider \( \overline{\gamma}^{\mu} \). Let’s keep in mind that \( \beta = \gamma^0 \) and that \( \gamma \)'s anticommute as \( \{ \gamma^\mu, \gamma^\nu \} = -2g^{\mu\nu} \). As Pauli matrices are Hermitian, \( \gamma^0 = \gamma^0 \) and \( \gamma^i = -\gamma^i \). Therefore, \( \overline{\gamma}^0 = \gamma^0 \) and \( \overline{\gamma}^i = -\gamma^i \). Further noting that \( \gamma^5 \) is Hermitian, and anticommutes with \( \gamma \)'s, \( \{ \gamma^\mu, \gamma^5 \} = 0 \), we proceed by considering \( i\gamma^5 \):

\[
\overline{\gamma}^5 : \quad \overline{i\gamma}^5 = -i\beta\gamma^5\beta = i\gamma^5 \\
\gamma^5 \gamma^5 : \quad \gamma^5 \gamma^5 = \beta\gamma^5\gamma^5 \beta = -\gamma^5 \gamma^5 = -\gamma^5 \gamma^5 = \gamma^5 \gamma^5 \\
i\gamma^5 S^{\mu\nu} : \quad i\gamma^5 S^{\mu\nu} = -iS^{\mu\nu} \gamma^5 = iS^{\mu\nu} \gamma^5 = i\gamma^5 S^{\mu\nu},
\]

where in the last line we twice anticommute \( \gamma^5 \) past the \( \gamma^\mu \)'s in \( S^{\mu\nu} \).

Problem 38.3

We start by recalling the equations of motion for four-component spinors

\[
(p + m)u_s(\vec{p}) = 0, \quad (-p + m)v_s(\vec{p}) = 0 \\
\bar{u}_s(\vec{p})(p + m) = 0, \quad \bar{v}_s(\vec{p})(-p + m) = 0
\]

and the convenient rewriting of \( \beta \) and \( \gamma^\mu \) orderings

\[
\gamma^\mu \beta = \frac{1}{2} \{ \gamma^\mu, \beta \} = -p^\mu - 2is^{\mu\nu} p_\nu, \quad \beta \gamma^\mu = \frac{1}{2} \{ \gamma^\mu, \beta \} + \frac{1}{2} [p, \gamma^\mu] = -p^\mu + 2is^{\mu\nu} p_\nu.
\]

Just as in the text we ‘conjugated’ the combination \( (p\gamma^\mu + \gamma^\mu \beta) \) by \( u \)'s and \( v \)'s, here we might consider conjugating by \( \bar{u}_s(\vec{p}) \) and \( v_s(\vec{p}) \) to get the desired expression, we find

\[
\bar{u}_s(\vec{p})(p\gamma^\mu + \gamma^\mu \beta)v_s(-\vec{p}) = -m\bar{u}_s(\vec{p})\gamma^\mu v_s(-\vec{p}) + m\bar{u}_s(\vec{p})\gamma^\mu v_s(-\vec{p}) = 0.
\]

This vanishes, maybe to get something that doesn’t vanish we should consider the conjugation of \( (p\gamma^\mu - \gamma^\mu \beta) \) by \( \bar{u}_s(\vec{p}) \) and \( v_s(-\vec{p}) \). We should also be careful about the distinction between \( p \) and
Therefore, we find that RHS are zero. We want to derive the Gordon identities. We then conjugate the expression \( \bar{u}_{s'}(\vec{p}) (\gamma^\mu - \gamma^\mu p') v_s(-\vec{p}) \).

Making use of the equations of motion, we can write this as

\[
\bar{u}_{s'}(\vec{p}) (\gamma^\mu - \gamma^\mu p') v_s(-\vec{p}) = -2m \bar{u}_{s'}(\vec{p}) \gamma^\mu v_s(-\vec{p}) .
\]

We can also make use of our convenient rewriting of \( \bar{p} \) and \( \gamma^\mu \)

\[
\bar{u}_{s'}(\vec{p}) (\gamma^\mu - \gamma^\mu p') v_s(-\vec{p}) = \bar{u}_{s'}(\vec{p}) (p' - p)^\mu + 2i S^\mu\nu (p + p'_\nu) v_s(-\vec{p}) ,
\]

and find that

\[
-2m \bar{u}_{s'}(\vec{p}) \gamma^\mu v_s(-\vec{p}) = \bar{u}_{s'}(\vec{p}) (p' - p)^\mu + 2i S^\mu\nu (p + p'_\nu) v_s(-\vec{p}) .
\]

The i components of this equation just reproduce what we should expect form the equations of motion, but the 0 component might give something interesting

\[
-2m \bar{u}_{s'}(\vec{p}) \gamma^0 v_s(-\vec{p}) = \bar{u}_{s'}(\vec{p}) \left((p_0' - p_0) + 2i(-S^{00}(p_0 + p'_0) + S^{0i}(\vec{p} - \vec{p})_i)\right) v_s(-\vec{p}) .
\]

As \( p^2 = p'^2 = -m^2 \), then \( p_0 = p'_0 \), and as \( S^{\mu\nu} \) is antisymmetric \( S^{00} = 0 \), meaning all terms on the RHS are zero

\[
\bar{u}_{s'}(\vec{p}) \gamma^0 v_s(-\vec{p}) = 0 .
\]

Equivalently, consider the conjugation of \( (\gamma^\mu - \gamma^\mu p') \) by \( \bar{v}_{s'}(\vec{p}) \) and \( u_s(-\vec{p}) \)

\[
\bar{v}_{s'}(\vec{p}) (\gamma^\mu - \gamma^\mu p') u_s(-\vec{p}) = 2m \bar{v}_{s'}(\vec{p}) \gamma^\mu u_s(-\vec{p}) = \bar{v}_{s'}(\vec{p}) (p' - p)^\mu + 2i S^\mu\nu (p + p'_\nu) u_s(-\vec{p}) ,
\]

for which the 0-th component tells us that

\[
\bar{v}_{s'}(\vec{p}) \gamma^0 u_s(-\vec{p}) = 0 .
\]

**Problem 38.4**

We want to derive the Gordon identities. We then conjugate the expression \( p' \gamma^\mu + \gamma^\mu \bar{p} \) with \( \bar{u}_{s'}(\vec{p}') \) and \( \gamma 5 u_s(\vec{p}) \)

\[
\bar{u}_{s'}(\vec{p}') (p' \gamma^\mu + \gamma^\mu \bar{p}) \gamma 5 u_s(\vec{p}) = \bar{u}_{s'}(\vec{p}') \left(- (p + p')^\mu + 2i S^\mu\nu (p - p'_\nu)\right) \gamma 5 u_s(\vec{p}) .
\]

Using the anticommutation of \( \gamma^5 \) and plugging in the Dirac equation for the spinors, we find

\[
\text{LHS} = \left( \bar{u}_{s'}(\vec{p}') \right) \gamma^\mu \gamma 5 u_s(\vec{p}) - \bar{u}_{s'}(\vec{p}') \gamma^\mu \gamma 5 \left( \bar{p} u_s(\vec{p})\right) = -m \bar{u}_{s'}(\vec{p}') \gamma^\mu \gamma 5 u_s(\vec{p}) + m \bar{u}_{s'}(\vec{p}') \gamma^\mu \gamma 5 u_s(\vec{p}) = 0 .
\]

Likewise, conjugating \( p' \gamma^\mu + \gamma^\mu \bar{p} \) with \( \bar{v}_{s'}(\vec{p}') \) and \( \gamma 5 v_s(\vec{p}) \)

\[
\bar{v}_{s'}(\vec{p}') (p' \gamma^\mu + \gamma^\mu \bar{p}) \gamma 5 v_s(\vec{p}) = \bar{v}_{s'}(\vec{p}') \left(- (p + p')^\mu + 2i S^\mu\nu (p - p'_\nu)\right) \gamma 5 v_s(\vec{p})
\]

\[
\text{LHS} = \left( \bar{v}_{s'}(\vec{p}') \right) \gamma^\mu \gamma 5 v_s(\vec{p}) - \bar{v}_{s'}(\vec{p}') \gamma^\mu \gamma 5 \left( \bar{p} v_s(\vec{p})\right) = m \bar{v}_{s'}(\vec{p}') \gamma^\mu \gamma 5 v_s(\vec{p}) - m \bar{v}_{s'}(\vec{p}') \gamma^\mu \gamma 5 v_s(\vec{p}) = 0 .
\]

Therefore, we find that

\[
\{ \bar{u}_{s'}(\vec{p}'), \bar{v}_{s'}(\vec{p}') \} (p + p')^\mu - 2i S^\mu\nu (p' - p)_\nu \gamma 5 \{ v_s(\vec{p}), v_s(\vec{p}) \} = 0 .
\]